Finite subset spaces of closed surfaces

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Abstract The kth finite subset space of a topological space X is the space $\exp_k X$ of non-empty finite subsets of X of size at most k, topologised as a quotient of X^k. The construction is a homotopy functor and may be regarded as a union of configuration spaces of distinct unordered points in X. We show that the finite subset spaces of a connected 2–complex admit "lexicographic cell structures" based on the lexicographic order on I^2 and use these to study the finite subset spaces of closed surfaces. We completely calculate the rational homology of the finite subset spaces of the two-sphere, and determine the top integral homology groups of $\exp_k \Sigma$ for each k and closed surface Σ. In addition, we use Mayer-Vietoris arguments and the ring structure of $H^*(\mathrm{Sym}^k \Sigma)$ to calculate the integer cohomology groups of the third finite subset space of Σ closed and orientable.

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1 Introduction

The kth finite subset space of a topological space X is the space $\exp_k X$ of nonempty subsets of X of size at most k, topologised as a quotient of X^k via the map sending each k–tuple to the set consisting of its entries. The construction is a homotopy functor, and if X is compact $\exp_k X$ may be regarded as a compactification of the configuration space of unordered k–tuples of distinct points in X.

In our previous papers [12, 13] we studied the finite subset spaces of the circle, connected graphs, and punctured surfaces. In this sequel we take the first steps towards an understanding of the finite subset spaces of closed surfaces. We do this in two orthogonal directions. We first use ideas and techniques developed in [12, 13] to show that finite subset spaces of connected 2–complexes

admit "lexicographic cell structures" based on the lexicographic ordering of I^2. Applying these to the standard cell structures for closed surfaces we completely calculate the rational homology of the finite subset spaces of S^2, and determine the top integral homology groups of $\exp_k \Sigma$ for each k and closed surface Σ. We then use a quite different approach to completely calculate the integral cohomology groups of the third finite subset space of a closed orientable surface. This is the first nontrivial case, since the second finite subset space co-incides with the second symmetric product, for which the answer is already known. We build a homotopy model for $\exp_3 \Sigma$ out of $\mathrm{Sym}^3 \Sigma$ and the mapping cylinder of $\Sigma^2 \to \mathrm{Sym}^2 \Sigma$, and calculate the cohomology using the Mayer-Vietoris sequence and the calculation of $H^*(\mathrm{Sym}^k \Sigma)$ due to Macdonald [8] and Seroul [10, 11].

1.1   Finite subset spaces

We begin by recalling some basic facts and constructions on finite subset spaces. For a history and bibliography see [13].

The kth finite subset space of X may be viewed as the quotient of the kth symmetric product $\mathrm{Sym}^k X = X^k / S_k$ obtained by forgetting multiplicities. The constructions co-incide for $k = 1$ and 2 but differ for $k \geq 3$, since the points (a, a, b) and (a, b, b) in X^3 map to distinct points in $\mathrm{Sym}^3 X$ but the same point in $\exp_3 X$. In particular $\exp_1 X = X$, $\exp_2 X = \mathrm{Sym}^2 X$, and $\exp_k X$ is a proper quotient of $\mathrm{Sym}^k X$ for $k \geq 3$.

A second view of $\exp_k X$ is obtained by regarding it as a union of configuration spaces of unordered tuples of distinct points in X. For $j \leq k$ there is a natural inclusion map

$$\exp_j X \hookrightarrow \exp_k X : \Lambda \mapsto \Lambda, \tag{1.1}$$

and if X is Hausdorff this is a homeomorphism onto its image [5]. In this case each stratum $\exp_j X \setminus \exp_{j-1} X$ is homeomorphic to the configuration space of unordered j–tuples of distinct points in X, so that $\exp_k X$ may be regarded as a union of these spaces, with the topology recognising that configurations of different cardinalities may be considered close. We define the full finite subset space $\exp X$ to be the direct limit of the system (1.1) of inclusions,

$$\exp X = \varinjlim \exp_k X.$$

Both $\exp_k$ and $\exp$ may be made into functors in the obvious way: given $f \colon X \to Y$ we define $\exp_k f$ and $\exp f$ by sending $\Lambda \subseteq X$ to $f(\Lambda) \subseteq Y$. The homotopy classes of $\exp_k f$ and $\exp f$ depend only on the homotopy class of f, making both $\exp_k$ and $\exp$ homotopy functors.

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Given a basepoint $x_0 \in X$ we define the kth based finite subset space to be the subspace

$$\exp_k(X, x_0) = \{\Lambda \in \exp_k X | x_0 \in \Lambda\}.$$

This subspace is the image of the map $\cup \{x_0\}$ that adjoins x_0 to each element of $\exp_{k-1} X$,

$$\cup \{x_0\}\colon \exp_{k-1} X \to \exp_k X : \Lambda \mapsto \Lambda \cup \{x_0\}.$$

It should be noted that $\exp_{k-1} X$ and $\exp_k(X, x_0)$ are in general topologically different, as $\cup \{x_0\}$ is generically two-to-one on the subspace $\exp_{k-2} X$ of $\exp_{k-1} X$. The based finite subset spaces $\exp_k(X, x_0)$ are often more tractable than the unbased spaces, and frequently play an important role as stepping stones to understanding them.

For each k and ℓ the isomorphism $X^k \times X^\ell \to X^{k+\ell}$ descends to a map

$$\cup\colon \exp_k X \times \exp_\ell X \to \exp_{k+\ell} X$$

sending (Λ_1, Λ_2) to $\Lambda_1 \cup \Lambda_2$. This leads to a form of product on maps $g\colon Y \to \exp_k X$, $h\colon Z \to \exp_\ell X$, and we define $g \cup h\colon Y \times Z \to \exp_{k+\ell} X$ to be the composition

$$Y \times Z \xrightarrow{g \times h} \exp_k X \times \exp_\ell X \xrightarrow{\cup} \exp_{k+\ell} X.$$

We will make extensive use of this product in constructing our cell structures for finite subset spaces, and we note that $(f \cup g) \cup h = f \cup (g \cup h)$.

1.2   Summary of main results

In this section and elsewhere in the paper we adopt the convention that where a co-efficient group or ring is not specified integer co-efficients should be assumed.

We first show that the finite subset spaces of a connected finite 2–complex admit lexicographic cell structures in section 2, and then we use these in section 3 to completely calculate the rational homology of $\exp_k(S^2, *)$ and $\exp_k S^2$.

Theorem 1   *The space $\exp_k(S^2, *)$ has the rational homology of S^{2k-2}, and the space $\exp_k S^2$ has the rational homology of $S^{2k} \vee S^{2k-2}$.*

More careful attention at the top end of the chain complex shows that "rational" cannot be replaced by "integral" in this theorem for $k \geq 4$ in the based case and $k \geq 3$ in the unbased case:

Theorem 2 *The top three integral homology groups of $\exp_k(S^2, *)$ are $\mathbf{Z}$ in dimension $2k-2$, $\{0\}$ in dimension $2k-3$, and $\mathbf{Z}/(k-2)\mathbf{Z}$ in dimension $2k-4$. The group H_{2k-4} is generated by the top homology class of $\exp_{k-1}(S^2, *) \hookrightarrow \exp_k(S^2, *)$.*

The top three integral homology groups of $\exp_k S^2$ are $\mathbf{Z}$ in dimension $2k$, $\{0\}$ in dimension $2k-1$, and $\mathbf{Z} \oplus \mathbf{Z}/(k-1)\mathbf{Z}$ in dimension $2k-2$. The group H_{2k-2} is generated by the top classes $[\exp_{k-1} S^2]$ of $\exp_{k-1} S^2$ and $[\exp_k(S^2, *)]$ of $\exp_k(S^2, *)$, subject to the relation $(k-1)\big([\exp_{k-1} S^2] - 2[\exp_k(S^2, *)]\big) = 0$.

In addition, we calculate the integral homology completely as far as $k = 6$ in the based case and $k = 5$ in the unbased case.

We then turn to the finite subset spaces of higher genus surfaces in section 4. Handel [5] has shown that $H^{nk}(\exp_k M^n; \mathbf{Z}/2\mathbf{Z})$ has rank one for a closed connected n–manifold, $n \geq 2$, and using the cell structures constructed in section 2 we prove the following refinement of this result for $n = 2$:

Theorem 3 *Let Σ be a closed surface of genus g. Then*

$$H_{2k}(\exp_k \Sigma) = \begin{cases} \mathbf{Z} & \text{if } \Sigma \text{ is orientable,} \\ 0 & \text{if } \Sigma \text{ is non-orientable,} \end{cases}$$

and

$$H_{2k-1}(\exp_k \Sigma) = \begin{cases} \mathbf{Z}^{2g} & \text{if } \Sigma \text{ is orientable,} \\ \mathbf{Z}/2\mathbf{Z} & \text{if } \Sigma \text{ is non-orientable.} \end{cases}$$

Given an orientation on Σ we may canonically orient the manifold $\exp_k \Sigma \setminus \exp_{k-1} \Sigma$ by orienting each tangent space as the direct sum

$$T_\Lambda(\exp_k \Sigma \setminus \exp_{k-1} \Sigma) = \bigoplus_{p \in \Lambda} T_p \Sigma. \tag{1.2}$$

Since Σ is even dimensional the order of the summands does not matter and we obtain a consistent orientation on the configuration space $\exp_k \Sigma \setminus \exp_{k-1} \Sigma$. This in turn orients H_{2k}, and Theorem 3 then implies that a map between the kth finite subset spaces of two closed oriented surfaces has a well defined degree. In the case where the map is induced by a map between the underlying spaces the degree behaves entirely as we might expect:

Theorem 4 *If $f\colon \Sigma \to \Sigma'$ is a map between closed oriented surfaces then*

$$\deg \exp_k f = (\deg f)^k.$$

Recall from [12] that the corresponding result for the circle is $\deg \exp_k f = (\deg f)^{\lfloor \frac{k+1}{2} \rfloor}$. Both results are proved in the same way, by counting preimages of a generic point in the target with sign, and the difference lies in the fact that the circle is odd-dimensional. Orienting tangent spaces to $\exp_k S^1 \setminus \exp_{k-1} S^1$ as in (1.2) requires the additional data of an order on the summands, and this leads to cancellation among the preimages, whereas no such cancellation occurs for surfaces.

As our last result for closed surfaces we construct a homotopy model for the third finite subset space of a closed orientable surface in section 5, and use this to completely calculate the integral cohomology groups of its third finite subset space in section 6.

Theorem 5 *Let Σ_g be a closed orientable surface of genus g. The cohomology group $H^i(\exp_3 \Sigma_g)$ is given by*

		g	
	0	1	≥ 2
0	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$
1	0	0	0
2	0	$\mathbf{Z}$	$\mathbf{Z}^{\binom{2g}{2}}$
$i \quad 3$	0	$\mathbf{Z}^5$	$\mathbf{Z}^{p(g)}$
4	$\mathbf{Z}$	$\mathbf{Z}^4$	$\mathbf{Z}^{q(g)} \oplus [\mathbf{Z}/2\mathbf{Z}]^{2g}$
5	$\mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}^2 \oplus \mathbf{Z}/2\mathbf{Z}$	$\mathbf{Z}^{2g} \oplus \mathbf{Z}/2\mathbf{Z}$
6	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$

in which

$$p(g) = \binom{2g}{3} + \binom{2g}{2} + \binom{2g}{1} \qquad \text{and} \qquad q(g) = \binom{2g}{2} + 1$$

for $g \geq 2$. The Euler characteristic is

$$\chi(\exp_3 \Sigma) = \frac{-4g^3 + 12g^2 - 17g + 9}{3}.$$

In particular $\chi(\exp_3 S^2) = 3$, $\chi(\exp_3 T^2) = 0$, and $\chi(\exp_3 \Sigma_2) = -3$.

2 Lexicographic cell structures

We show that the finite subset spaces of connected finite 2-complexes admit cell structures based on the lexicographic ordering of I^2. After recalling some

facts about finite subset spaces of graphs and establishing some conventions in section 2.1 we prove the cell structures exist in section 2.2, and calculate the boundary maps in section 2.3. We close in section 2.4 with a brief discussion on constructing lexicographic cell structures for the finite subset spaces of higher dimensional complexes.

We remark that the existence of lexicographic cell structures is chiefly of practical rather than theoretical importance, as the finite subset spaces of a simplicial complex may be shown to have cell structures using the machinery of simplicial sets (Jacob Mostovoy, private communication). Simplicial sets are described in May's book [9] and Curtis's article [2]. Given a simplicial set K we let $\exp_j K$ be the simplicial set whose n–simplices are subsets of size at most j of the n–simplices of K, and whose face and degeneracy operators are the face and degeneracy operators of K acting elementwise. Then if X is the geometric realisation of K, $\exp_j X$ will be the geometric realisation of $\exp_j K$, showing that triangulated spaces have triangulated finite subset spaces. The power of this method comes at the expense of difficulties with concrete calculations: for example, the triangulations produced for $\exp_3 S^2$ and $\exp_4 S^2$ have 77 and 1039 cells,[1] in contrast with the corresponding lexicographic cell structures which have 11 and 23.

Although their existence is not of immediate theoretical interest, the construction of lexicographic cell structures for the finite subset spaces of connected 2–complexes does have an interesting theoretical consequence. Up to homotopy we may assume that X has a single vertex, and then the lexicographic cell structure for $\exp_k X$ is obtained from that of $\exp_{k-1} X$ by adding cells in dimensions $k - 1 \leq d \leq 2k$, or $k \leq d \leq 2k$ if X has no edges. Using a result of Handel [5] this implies that $\exp_k X$ is $(k-2)$–connected, and $(k-1)$–connected if X is simply connected. In [14] we use this to show that the same is true without the dimension or finiteness restrictions.

2.1 Conventions and definitions

In constructing cell structures for the finite subset spaces of a connected finite 2–complex X we will work up to homotopy and assume that X has a single vertex, as we did with graphs in [13]. There we let Γ_n be the graph with a single vertex v and n edges $e_1, \ldots, e_n$, and here we will identify the 1–skeleton of X with Γ_n for some $n \geq 0$ and denote its 2–cells by $f_1, \ldots, f_m$. Each cell

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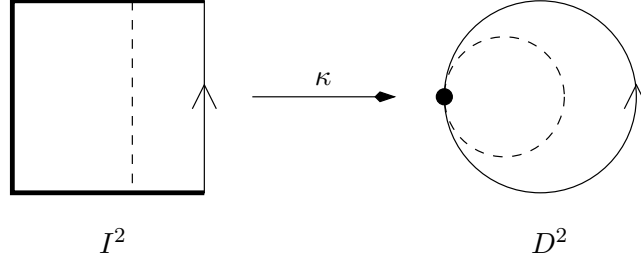[1]These numbers are derived from cell counts sent by Jacob Mostovoy.

Figure 1: The crunching map $\kappa\colon I^2 \to D^2$. The three sides of the square shown in bold map to the dot at -1, the unbolded edge maps to the boundary, and the dashed line goes to the dashed circle.

f_i has a characteristic map ϕ_i from the unit disc $D^2 \subseteq \mathbf{C}$ to X and we assume that ϕ_i sends $-1 \in \partial D^2$ to v.

By Lemma 1 of [13] $\exp_k \Gamma_n$ has a cell structure consisting of cells σ^J, $\tilde{\sigma}^J$ in which the indices are n–tuples $J = (j_1, \ldots, j_n)$ of non-negative integers. The cells σ^J form a cell structure for $\exp_k(\Gamma_n, v)$, and the indexing n–tuple of the cell σ^J or $\tilde{\sigma}^J$ containing the subset Λ in its interior is determined by the integers $j_i = |\Lambda \cap \operatorname{int} e_i|$. Letting $\tilde{\Delta}_j$ be the simplex

$$\tilde{\Delta}_j = \{(x_1, \ldots, x_j)|0 \le x_1 \le \cdots \le x_j \le 1\},$$

the domain of σ^J, $\tilde{\sigma}^J$ is the ball $\tilde{\Delta}_J = \tilde{\Delta}_{j_1} \times \cdots \times \tilde{\Delta}_{j_n}$, in which we omit any empty factors $\tilde{\Delta}_0$.

The cell structures for $\exp_k \Gamma_n$ were constructed by using the linear order on each edge to choose preferred lifts from $\exp_j \Gamma_n$ to $(\Gamma_n)^j$. To build cell structures for $\exp_k X$ we proceed similarly using the lexicographic order $\prec$, defined on the unit square I^2 by

$$(x_1, y_1) \prec (x_2, y_2) \qquad \text{if } x_1 < x_2, \text{ or if } x_1 = x_2 \text{ and } y_1 < y_2.$$

We transfer this order to $\operatorname{int} D^2$ via a "crunching" map $\kappa\colon I^2 \to D^2$ which sends the three sides $x = 0$, $y = 0$ and $y = 1$ to $-1 \in \partial D^2$ and maps the rest of the square homeomorphically onto the rest of the disc, preserving orientation. The exact form of κ is unimportant, but for the sake of concreteness we let

$$\kappa(x, y) = x(1 - e^{2\pi i y}) - 1.$$

This sends the vertical segment $x = x_0$ to the circle with centre $x_0 - 1$ and radius x_0, as shown in figure 1.

As was the case with graphs our cells will be indexed by vectors of integers corresponding to ordered partitions, but in contrast with graphs the vectors

arising from a 2–cell will consist of positive integers only and their lengths will vary. We introduce some notation and terminology that will be helpful in working with such partitions.

For each non-negative integer m let $[m] = \{i \in \mathbf{Z} | 1 \leq i \leq m\}$. Given a vector S of positive or non-negative integers we write $\ell(S) = \ell$ if $S = (s_1, \ldots, s_\ell)$, and we define the norm of S to be

$$|S| = \sum_{i=1}^{\ell(S)} s_i.$$

For each subset α of $[\ell(S)]$ we write $S|_\alpha$ for the $|\alpha|$–tuple obtained by restricting the index set to α.

The boundary of the cell corresponding to S will consist of two main contributions, one a sum of cells corresponding to partitions obtained by merging adjacent parts of S, and the second a sum of cells corresponding to partitions obtained by decreasing a part of S by one. Accordingly we define

$$\mu_i(S) = (s_1, \ldots, s_i + s_{i+1}, \ldots, s_\ell)$$

for $1 \leq i \leq \ell(S) - 1$, and

$$\partial_i(S) = (s_1, \ldots, s_i - 1, \ldots, s_\ell)$$

for $1 \leq i \leq \ell(S)$. Notice that μ_i decreases the length of S by one, while ∂_i decreases the norm of S by one.

Finally, the length-decreasing contribution to the boundary will involve the (-1)–binomial co-efficient $\begin{bmatrix} m \\ r \end{bmatrix}_{-1}$, which we recall is a signed count of the ways of choosing r elements from the set $[m]$. Each choice is counted with the sign of the permutation obtained by ordering the set $[m]$ so that the chosen elements occur first in ascending order, followed by the unchosen elements in ascending order. The value of $\begin{bmatrix} m \\ r \end{bmatrix}_{-1}$ was calculated in section 3.2 of [13] and is given by

$$\begin{bmatrix} m \\ r \end{bmatrix}_{-1} = \frac{1 + (-1)^{r(m-r)}}{2} \left(\begin{matrix} \lfloor m/2 \rfloor \\ \lfloor r/2 \rfloor \end{matrix} \right).$$

2.2 Existence of lexicographic cell structures

The first step in building a cell structure for $\exp_k X$ is to form an open cell decomposition of $\exp_j(\text{int } f_i) \setminus \exp_{j-1}(\text{int } f_i)$, using the fact that each j element subset of $\text{int } I^2$ has a unique lexicographically ordered representative in $(\text{int } I^2)^j$. Figure 2 illustrates the idea for $k = 3$. Generically, three points in
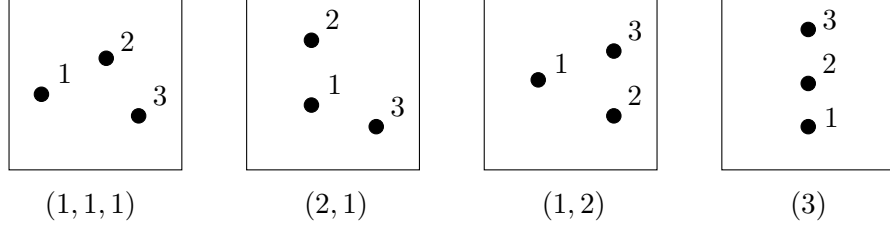
Figure 2: Open cell decomposition of $\exp_3(\operatorname{int} I^2) \setminus \exp_2(\operatorname{int} I^2)$. We decompose $\exp_3(\operatorname{int} I^2) \setminus \exp_2(\operatorname{int} I^2)$ as a union of four open cells, corresponding to the four ordered partitions $(1,1,1)$, $(2,1)$, $(1,2)$ and (3) of three as a sum of positive integers.

int I^2 may be ordered by their x co-ordinates, as in the square labeled $(1,1,1)$, and this gives an open 6–ball

$$\operatorname{int}(\tilde{\Delta}_3 \times \tilde{\Delta}_1 \times \tilde{\Delta}_1 \times \tilde{\Delta}_1) = \{0 < x_1 < x_2 < x_3 < 1\} \times \{0 < y_1, y_2, y_3 < 1\}.$$

If two of the x co-ordinates are equal but the third is different there are two possibilities, illustrated by the squares labeled $(2,1)$ and $(1,2)$ and corresponding to the open 5-balls

$$\operatorname{int}(\tilde{\Delta}_2 \times \tilde{\Delta}_2 \times \tilde{\Delta}_1)$$
$$= \{0 < x_1 = x_2 < x_3 < 1\} \times \{0 < y_1 < y_2 < 1\} \times \{0 < y_3 < 1\},$$

$$\operatorname{int}(\tilde{\Delta}_2 \times \tilde{\Delta}_1 \times \tilde{\Delta}_2)$$
$$= \{0 < x_1 < x_2 = x_3 < 1\} \times \{0 < y_1 < 1\} \times \{0 < y_2 < y_3 < 1\},$$

and when all three x co-ordinates are equal as in the square labeled (3) we have the open 4–ball

$$\operatorname{int}(\tilde{\Delta}_1 \times \tilde{\Delta}_3) = \{0 < x_1 = x_2 = x_3 < 1\} \times \{0 < y_1 < y_2 < y_3 < 1\}.$$

This gives a decomposition of $\exp_3(\operatorname{int} I^2) \setminus \exp_2(\operatorname{int} I^2)$ as a union of four open cells, and more generally we obtain a decomposition of $\exp_j(\operatorname{int} I^2) \setminus \exp_{j-1}(\operatorname{int} I^2)$ as a union of 2^{j-1} open cells, indexed by the ordered partitions of j as a sum of positive integers. A cell structure for $\exp_k X$ is then obtained by taking products of such cells with the cells $\sigma^J, \tilde{\sigma}^J$ of $\exp_k \Gamma_n$.

Concretely, to each j element subset $\Lambda = \{p_1, \ldots, p_j\}$ of int I^2 we associate the ordered partition $\mathcal{S}(\Lambda) = (s_1, \ldots, s_\ell)$ of j arising from the equivalence relation $p_i \sim p_j$ if $x_i = x_j$. This gives a partition $\{\Lambda_1, \ldots, \Lambda_\ell\}$ of Λ which may be ordered by $\Lambda_a < \Lambda_b$ if $p \in \Lambda_a$, $p' \in \Lambda_b$, and $x < x'$, and we obtain $\mathcal{S}(\Lambda)$ by letting s_i be the size of the ith part. Set

$$\mathcal{E}(S) = \{\Lambda \in \exp_{|S|}(\operatorname{int} I^2) | \mathcal{S}(\Lambda) = S\}$$

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for each vector S of positive integers. It is clear that the sets $\mathcal{E}(S)$ with $|S| = j$ form a partition of $\exp_j(\operatorname{int} I^2) \setminus \exp_{j-1}(\operatorname{int} I^2)$, and we claim further that $\mathcal{E}(S)$ is parameterised by the open $(|S| + \ell(S))$–ball $\operatorname{int}(\tilde{\Delta}_S \times \tilde{\Delta}_{\ell(S)})$. To see this, note that if $\{p_1, \ldots, p_j\} \in \mathcal{E}(S)$ satisfies $p_1 \prec \cdots \prec p_j$ then $(y_1, \ldots, y_j)$ is a point in $\operatorname{int}(\tilde{\Delta}_{s_1} \times \cdots \times \tilde{\Delta}_{s_\ell})$ and the distinct x values $(x_{i_1}, \ldots, x_{i_\ell})$ form a point in $\operatorname{int} \tilde{\Delta}_{\ell(S)}$. This leads to a map

$$\operatorname{int}(\tilde{\Delta}_{s_1} \times \cdots \times \tilde{\Delta}_{s_\ell} \times \tilde{\Delta}_\ell) \to (\operatorname{int} I^2)^j \to \exp_j(\operatorname{int} I^2) \qquad (2.1)$$

hitting precisely $\mathcal{E}(S)$, and this map is injective since each $\Lambda \in \exp_j(\operatorname{int} I^2) \setminus \exp_{j-1}(\operatorname{int} I^2)$ has a unique lexicographically ordered representative in $(\operatorname{int} I^2)^j$.

Having found an open cell decomposition of $\exp_j(\operatorname{int} I^2)$ we now construct cells for $\exp_k X$. The map (2.1) extends to a map $\tilde{\Delta}_S \times \tilde{\Delta}_\ell \to \exp_j I^2$ and we let $\tilde{\tau}_i^S$ be the composition

$$\tilde{\Delta}_S \times \tilde{\Delta}_{\ell(S)} \to \exp_{|S|} I^2 \xrightarrow{\exp_{|S|} \kappa} \exp_{|S|} D^2 \xrightarrow{\exp_{|S|} \phi_i} \exp_{|S|} f_i,$$

τ_i^S the composition

$$\tilde{\Delta}_S \times \tilde{\Delta}_{\ell(S)} \xrightarrow{\tilde{\tau}_i^S} \exp_{|S|} f_i \xrightarrow{\cup\{v\}} \exp_{|S|+1}(f_i, v).$$

Since both $\kappa|_{\operatorname{int} I^2}$ and $\phi_i|_{\operatorname{int} D^2}$ are homeomorphisms so are $\exp_k(\kappa|_{\operatorname{int} I^2})$ and $\exp_k(\phi_i|_{\operatorname{int} D^2})$, and it follows that the maps $\tilde{\tau}_i^S$ are injective on the interiors of their balls of definition and give an open cell decomposition of $\exp_j(\operatorname{int} f_i)$. Further, τ_i^S is injective on $\operatorname{int}(\tilde{\Delta}_S \times \tilde{\Delta}_{\ell(S)})$ as well, since $\cup\{v\}$ is injective on $\exp_j(\operatorname{int} f_i)$. We claim:

Theorem 6 *Let X be a connected finite 2–complex with a single vertex v. Then the maps*

$$\left\{ \tau_{i_1}^{S_1} \cup \cdots \cup \tau_{i_p}^{S_p} \cup \sigma^J \,|\, i_1 < \cdots < i_r \text{ and } |J| + \sum_q |S_q| \le k - 1 \right\} \qquad (2.2)$$

are the characteristic maps of a cell structure for $\exp_k(X, v)$, and these maps together with

$$\left\{ \tilde{\tau}_{i_1}^{S_1} \cup \cdots \cup \tilde{\tau}_{i_p}^{S_p} \cup \tilde{\sigma}^J \,|\, i_1 < \cdots < i_r \text{ and } |J| + \sum_q |S_q| \le k \right\} \qquad (2.3)$$

are the characteristic maps of a cell structure for $\exp_k X$.

Notice that $\exp_k X$ is obtained from $\exp_{k-1} X$ by adding cells in dimensions $k - 1 \le d \le 2k$, or $k \le d \le 2k$ if X has no edges. Handel [5] has shown that for path connected Hausdorff Y the inclusion map $\exp_k Y \hookrightarrow \exp_{2k+1} Y$ is zero on all homotopy groups, and together these results imply that $\exp_k X$ is $(k-2)$–connected, and $(k-1)$–connected if X is simply connected. We show that this holds more generally for higher dimensional complexes in [14].

Proof Each map $\tau_{i_1}^{S_1} \cup \cdots \cup \tau_{i_p}^{S_p} \cup \sigma^J$ in (2.2) is a map from a $|J| + \sum_q |S_q| + \sum_q \ell(S_q)$ dimensional ball to $\exp_{(|J| + \sum_q |S_q| + 1)}(X, v)$, and since $|J| + \sum_q |S_q| \leq k - 1$ the image of each map lies in $\exp_k(X, v)$. Similarly, each map $\tilde{\tau}_{i_1}^{S_1} \cup \cdots \cup \tilde{\tau}_{i_p}^{S_p} \cup \tilde{\sigma}^J$ occurring in (2.3) is a map from a ball of dimension $|J| + \sum_q |S_q| + \sum_q \ell(S_q)$ with image lying in $\exp_k X$, and moreover the interior of this ball misses $\exp_k(X, v)$. We show that each map is injective on the interior of its domain, that the open cells partition $\exp_k X$, and that the boundary of each cell lies on open cells of strictly smaller dimension.

To prove injectivity on interiors note that each basic cell $\tilde{\tau}_{i_q}^{S_q}$, $\tilde{\sigma}_i^{j_i}$ is injective on the interior of its domain of definition, and that the images of these cells are disjoint since they lie in disjoint sets $\exp_{|S_q|}(\text{int } f_{i_q})$, $\exp_{j_i}(\text{int } e_i)$. It follows that the restriction of the cupped map $\tilde{\tau}_{i_1}^{S_1} \cup \cdots \cup \tilde{\tau}_{i_p}^{S_p} \cup \tilde{\sigma}^J$ to the interior of its ball of definition is injective also. Further, as noted above this restriction misses $\exp_k(X, v)$, implying that

$$\tau_{i_1}^{S_1} \cup \cdots \cup \tau_{i_p}^{S_p} \cup \sigma^J = (\cup\{v\}) \circ (\tilde{\tau}_{i_1}^{S_1} \cup \cdots \cup \tilde{\tau}_{i_p}^{S_p} \cup \tilde{\sigma}^J)$$

is injective on the interior of its ball of definition also.

Each $\Lambda \in \exp_k X$ can be written uniquely as

$$\Lambda = (\Lambda \cap \{v\}) \cup \bigcup_{i=1}^{n} (\Lambda \cap (\text{int } e_i)) \cup \bigcup_{i=1}^{m} (\Lambda \cap (\text{int } f_i))$$

and this data determines a unique open cell from (2.2) or (2.3) containing Λ. Let $i_1 < \cdots < i_p$ be the indices i such that $\Lambda \cap (\text{int } f_i)$ is non-empty, set $S_q = \mathcal{S}(\Lambda \cap (\text{int } f_{i_q}))$, and let $j_i = |\Lambda \cap (\text{int } e_i)|$. Then the open cell $\tau_{i_1}^{S_1} \cup \cdots \cup \tau_{i_p}^{S_p} \cup \sigma^J$ contains Λ if $v \in \Lambda$, and the open cell $\tilde{\tau}_{i_1}^{S_1} \cup \cdots \cup \tilde{\tau}_{i_p}^{S_p} \cup \tilde{\sigma}^J$ contains Λ if $v \notin \Lambda$. Moreover, this is the unique open cell containing Λ: regarding $\tau_{i_1}^{S_1} \cup \cdots \cup \tau_{i_p}^{S_p} \cup \sigma^J$ as $\tilde{\tau}_{i_1}^{S_1} \cup \cdots \cup \tilde{\tau}_{i_p}^{S_p} \cup \tilde{\sigma}^J \cup \{v\}$, the map $\tilde{\tau}_i^S$ can occur as a factor if and only if $\Lambda \cap (\text{int } f_i)$ is non-empty and $S = \mathcal{S}(\Lambda \cap (\text{int } f_i))$, $\tilde{\sigma}^J$ can occur as a factor if and only if $j_i = |\Lambda \cap (\text{int } e_i)|$ for each i, and $\{v\}$ can occur as a factor if and only if $v \in \Lambda$. It follows that the open cells (2.2) and (2.3) partition $\exp_k X$ as claimed.

It remains to check that taking boundaries decreases dimension. It will suffice to do this for each factor cell, and since we know the result for $\tilde{\sigma}^J, \sigma^J$ by Lemma 1 of [13] it will be enough to check the cells τ_i^S, $\tilde{\tau}_i^S$. We do this in section 2.3, where we calculate $\partial \tau_i^S$ and $\partial \tilde{\tau}_i^S$. $\qquad \square$

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2.3 The boundary maps

In this section we indicate how to calculate the boundaries of the cells (2.2) and (2.3), and complete the proof of Theorem 6 by showing that taking boundaries decreases dimension. Since

$$\partial(\sigma \cup \tau) = \partial(\cup \circ (\sigma \times \tau))$$
$$= \cup_\sharp \partial(\sigma \times \tau)$$
$$= \cup_\sharp (\partial\sigma \times \tau + (-1)^{\dim \sigma} \sigma \times \partial\tau) \quad (2.4)$$

we need only calculate the boundary of each basic cell occurring as a factor and understand certain special cases of the cellular chain map $\cup_\sharp$. The only nontrivial case that will arise in this context is the case $\cup_\sharp(\tilde{\sigma}^J \times \tilde{\sigma}^L)$ studied in [13], so we will be able to confine our attention to calculating the boundaries of τ_i^S and $\tilde{\tau}_i^S$. To do this we calculate $\partial\tilde{\tau}_i^S$ and obtain $\partial\tau_i^S$ as $(\cup\{v\})_\sharp \partial\tilde{\tau}_i^S$ by removing all tildes.

The boundary of $\tilde{\tau}_i^S$ will consist of three parts, one that decreases the norm of S, a second that decreases the length of S, and a third that moves points from the interior of f_i onto the boundary. In preparation for stating the result we define each separately. Recall from section 2.1 that $\partial_a(S)$ is the partition obtained by decreasing the ath part of S by one, and that $\mu_a(S)$ is the partition obtained by merging the ath and $(a+1)$th parts of S, in other words

$$\partial_a(S) = (s_1, \ldots, s_a - 1, \ldots, s_\ell),$$
$$\mu_a(S) = (s_1, \ldots, s_a + s_{a+1}, \ldots, s_\ell).$$

Since $\partial_a(S)$ has norm $|S| - 1$ and length $\ell(S)$ the cell $\tilde{\tau}_i^{\partial_a(S)}$ has dimension one less than $\tilde{\tau}_i^S$, and the same is true of $\tilde{\tau}_i^{\mu_a(S)}$ since $|\mu_a(S)| = |S|$ and $\ell(\mu_a(S)) = \ell(S) - 1$. Let

$$\partial^\nu \tau_i^S = -\sum_{a=1}^{\ell(S)} \frac{1 + (-1)^{s_a}}{2} (-1)^{|S|_{[a-1]}|} \tau_i^{\partial_a(s)},$$

$$\partial^\lambda \tau_i^S = \sum_{a=1}^{\ell(S)-1} (-1)^{(a-1)} \begin{bmatrix} s_a + s_{a+1} \\ s_a \end{bmatrix}_{-1} \tau_i^{\mu_a(S)},$$

and extend these maps to $\tilde{\tau}_i^S$ by

$$\partial^\nu \tilde{\tau}_i^S = 2\partial^\nu \tau_i^S - \widetilde{\partial^\nu \tau_i^S},$$
$$\partial^\lambda \tilde{\tau}_i^S = \widetilde{\partial^\lambda \tau_i^S}.$$

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The Greek letters ν and λ are intended as mnemonics for "norm" and "length" respectively, and as with graphs a tilde over a chain τ means each cell τ_i^S in τ should be replaced with $\tilde{\tau}_i^S$. The linear combinations $\partial^\nu \tau_i^S$, $\partial^\nu \tilde{\tau}_i^S$ can be recognised as $\partial \sigma^S$, $\partial \tilde{\sigma}^S$ with each σ replaced by τ_i, and it follows that ∂^ν is a boundary operator. It is a consequence of the boundary calculation below with X specialised to S^2 that ∂^λ is a boundary operator as well, and that these two operators commute.

For the third contribution to the boundary of $\tilde{\tau}_i^S$ we identify ∂D^2 with Γ_1, sending v to -1 and giving ∂D^2 the anti-clockwise orientation. Writing $w_i \colon \Gamma_1 \to \Gamma_n$ for the attaching map of f_i, there are chain maps $(\exp_j w_i)_\sharp$ and we define

$$\partial^\gamma \tau_i^S = \tau_i^{S|_{[\ell-1]}} \cup (\exp_{s_\ell} w_i)_\sharp \sigma^{s_\ell},$$
$$\partial^\gamma \tilde{\tau}_i^S = \widetilde{\partial^\gamma \tau_i^S}.$$

Then:

Theorem 7 *The boundary of $\tilde{\tau}_i^S$ is given by*

$$\partial \tilde{\tau}_i^S = \partial^\nu \tilde{\tau}_i^S + (-1)^{|S|} \partial^\lambda \tilde{\tau}_i^S + (-1)^{|S|+\ell-1} \partial^\gamma \tilde{\tau}_i^S.$$

To calculate the boundary of a product cell $\tilde{\tau}_{i_1}^{S_1} \cup \cdots \cup \tilde{\tau}_{i_p}^{S_p} \cup \tilde{\sigma}^J$ use (2.4) and observe that the only nontrivial cases of $\cup_\sharp$ that occur come from the ∂^γ terms and have the form $\cup_\sharp (\tilde{\tau}_{i_1}^{S_1} \times \cdots \times \tilde{\sigma}^L \times \cdots \times \tilde{\tau}_{i_p}^{S_p} \times \tilde{\sigma}^J)$. Since $\sigma \times \tau = \pm \tau \times \sigma$ we need only understand the case $\cup_\sharp (\tilde{\sigma}^L \times \tilde{\sigma}^J)$, which is given by Section 3 of [13].

Proof of Theorem 7 The cell $\tilde{\tau}_i^S$ is a map from the ball $\tilde{\Delta}_{s_1} \times \cdots \times \tilde{\Delta}_{s_\ell} \times \tilde{\Delta}_\ell$ and we consider the effect of replacing an inequality with an equality in a simplex in this product. There are two main cases, the simplices $\tilde{\Delta}_{s_a}$ and $\tilde{\Delta}_\ell$, and each breaks into further cases according to whether the equality occurs in the first, last, or an inner position. Four of the possibilities are illustrated in figure 3.

Replacing the first or last inequality in $\tilde{\Delta}_{s_a}$ sends a point in the interior of I^2 onto the top or bottom edge of the square, as shown in figure 3(i). Since both of these edges map to v in X each case gives a point in $\tau_i^{\partial_a(S)}$, dropping dimension by one, unless $s_a = 1$ in which case it gives a point in $\tau_i^{S|_{[\ell]\setminus\{a\}}}$, dropping dimension by two. A generic point in $\tau_i^{\partial_a(S)}$ is hit once by each face,
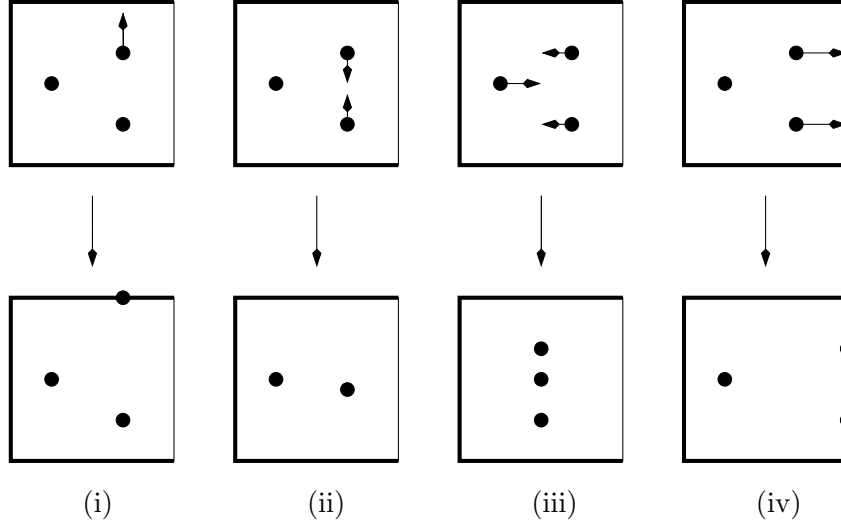
Figure 3: Some cells in the boundary of $\tilde{\tau}^{(1,2)}$. (i) Moving a point onto the top or bottom edge gives a set lying in $\tau^{(1,1)}$ (illustrated) or $\tau^{(2)}$ (not shown). (ii) Moving two points with the same x co-ordinate into co-incidence gives a point in $\tilde{\tau}^{(1,1)}$. (iii) Equating distinct x co-ordinates merges two columns, giving a point in $\tilde{\tau}^{(3)}$, and (iv) moving the rightmost column onto the boundary gives a point in $\tilde{\tau}^{(1)} \cup \tilde{\sigma}^2$.

and as was the case with graphs in [13] these contributions come with opposite signs if s_a is odd and matching signs if s_a is even, the sign negative if $a = 1$.

Replacing an inner inequality in $\tilde{\Delta}_{s_a}$ merges two points with the same x co-ordinate, giving a point in $\tilde{\tau}_i^{\partial_a(S)}$ as shown in figure 3(ii). A generic point in this cell is hit $s_a - 1$ times, once for each inequality, and as was the case with graphs these contributions alternate in sign. They therefore cancel if s_a is odd and leave one over if s_a is even, the leftover sign being positive if $a = 1$. Putting these together with the contributions to $\tau_i^{\partial_a(S)}$ from the previous paragraph we get the $\partial^\nu \tilde{\tau}_i^S$ term in $\partial \tilde{\tau}_i^S$.

We now look at the contributions to the boundary from $\tilde{\Delta}_\ell$. Replacing the first inequality with an equality moves the entire first column of points onto the left edge of the square, which maps to v in X. Thus this face maps onto $\tau_i^{S|_{[\ell] \setminus \{1\}}}$, dropping dimension by at least two and so contributing nothing to $\partial \tilde{\tau}_i^S$.

Replacing the ath inner inequality merges the ath and $(a+1)$th columns of points, giving a point in $\tilde{\tau}_i^{\mu_a(S)}$ as shown in figure 3(iii). Generic points in this cell are each hit $\binom{s_a + s_{a+1}}{s_a}$ times, once for each choice of s_a points to come from

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the left, and counting these with signs gives the (-1)–binomial co-efficient $\begin{bmatrix} s_a+s_{a+1} \\ s_a \end{bmatrix}_{-1}$. Collecting these contributions together gives the $\partial^\lambda \tilde{\tau}_i^S$ term in $\partial \tilde{\tau}_i^S$.

Finally, we look at the contribution to the boundary from the last face of $\tilde{\Delta}_\ell$, which moves the last column of points onto the right edge of the square, as shown in figure 3(iv). The points left in the interior of the square lie in the cell $\tilde{\tau}_i^{S|[\ell-1]}$, which has dimension $|S| + \ell - s_\ell - 1$, and the points on the boundary lie in some cell σ^J or $\tilde{\sigma}^J$ of $\exp_{s_\ell} \Gamma_n$, which has dimension s_ℓ. Thus this face maps onto cells of smaller dimensions also. The signed sum of the dimension s_ℓ cells hit in $\exp_{s_\ell} \Gamma_n$ is given by $(\exp_{s_\ell} w_i)_\sharp \tilde{\sigma}^{s_\ell}$, and we get the $\partial^\gamma \tilde{\tau}_i^S$ term of $\partial \tilde{\tau}_i^S$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The boundary map ∂^λ can be simplified by choosing a suitable basis for the chain groups over $\mathbf{Q}$, at the expense of making ∂^ν mildly more complicated. Recalling from section 2.1 that the (-1)–binomial co-efficient $\begin{bmatrix} m \\ r \end{bmatrix}_{-1}$ is given by

$$\begin{bmatrix} m \\ r \end{bmatrix}_{-1} = \frac{1 + (-1)^{r(m-r)}}{2} \begin{pmatrix} \lfloor m/2 \rfloor \\ \lfloor r/2 \rfloor \end{pmatrix}$$

we let

$$\tau_{\mathbf{Q}}^S = (\lfloor s_1/2 \rfloor! \cdots \lfloor s_\ell/2 \rfloor!)\tau^S.$$

Then

$$\begin{bmatrix} s_a+s_{a+1} \\ s_a \end{bmatrix}_{-1} \tau_{\mathbf{Q}}^{\mu_a(S)} = \frac{1 + (-1)^{s_a s_{a+1}}}{2} \frac{\lfloor (s_a + s_{a+1})/2 \rfloor!}{\lfloor s_a/2 \rfloor! \lfloor s_{a+1}/2 \rfloor!} (\lfloor s_1/2 \rfloor! \cdots \lfloor s_\ell/2 \rfloor!)\tau^{\mu_a(S)}$$

$$= \frac{1 + (-1)^{s_a s_{a+1}}}{2} \tau_{\mathbf{Q}}^{\mu_a(S)},$$

so that with respect to this basis we have

$$\partial^\lambda \tau_{\mathbf{Q}}^S = \sum_{a=1}^{\ell(S)-1} \frac{1 + (-1)^{s_a s_{a+1}}}{2} (-1)^{(a-1)} \tau_{\mathbf{Q}}^{\mu_a(S)}. \qquad (2.5)$$

This simplification comes at the small price of adding a factor of $\lfloor s_a/2 \rfloor$ to the ath term of ∂^ν when s_a is even, giving

$$\partial^\nu \tau_{\mathbf{Q}}^S = -\sum_{a=1}^{\ell(S)} \frac{1 + (-1)^{s_a}}{4} (-1)^{|S|_{[a-1]}|} s_a \tau_{\mathbf{Q}}^{\partial_a(s)}.$$

This mild complication in ∂^ν will not concern us, since our main use of (2.5) will be in calculating the rational homology of $\exp_k(S^2, *)/\exp_{k-1}(S^2, *)$, where the boundary map consists only of ∂^λ. We note further that $\tau_{\mathbf{Q}}^S = \tau^S$ if each entry of S is at most three.

2.4   Higher dimensions

We make a brief digression on constructing lexicographic cell structures for the finite subset spaces of higher dimensional complexes. As was the case with 2–complexes, the lexicographic ordering on I^n may be used to construct an open cell decomposition of $\exp_j(\operatorname{int} I^n)$, and we might hope to use this to build a cell structure for $\exp_k X$ by taking products over cells of X as before. However, ensuring that boundaries decrease dimension under this scheme appears to require compatibility between the ordering on the interior of a cell and the orderings on cells in its boundary. This condition comes for free in the case of 2–complexes or wedges of spheres but appears to require orchestration in general.

To each lexicographically ordered sequence $p_1, \ldots, p_j$ of j points in the interior of I^n we associate a $(j-1)$–tuple of integers by setting $j_i = m - 1$ if p_i and p_{i+1} are distinguished by their mth co-ordinate but not their $(m-1)$th. This $(j-1)$–tuple describes the dependencies among the co-ordinates of the p_i and as before we consider the collection of lexicographically ordered subsets corresponding to a fixed tuple J. This is easily seen to be parameterised by an open ball of dimension $nj - |J|$, a product of open simplices of the form $\operatorname{int} \tilde{\Delta}_a$, and this gives an open cell decomposition of $\exp_j(\operatorname{int} I^n)$. We again transfer this to $\exp_j(\operatorname{int} B^n)$ by collapsing the entire boundary to a point, with the exception of the sole face $x = 1$; the purpose of this is to avoid constraints between points in the interior and points on the boundary which would otherwise arise along faces of the form shown in figure 3(i).

To illustrate the compatibility requirement let $n = 3$ and give the face $x = 1$ the reverse lexicographic ordering

$$(1, y_1, z_1) \prec (1, y_2, z_2) \qquad \text{if } z_1 < z_2, \text{ or if } z_1 = z_2 \text{ and } y_1 < y_2.$$

Three element subsets of $\operatorname{int} I^3$ corresponding to the 2–tuple $(2, 2)$ satisfy $x_1 = x_2 = x_3$, $y_1 = y_2 = y_3$ and $z_1 < z_2 < z_3$, and are parameterised by the open 5–ball $\operatorname{int}(I \times I \times \tilde{\Delta}_3)$. Pushing them onto the boundary we obtain subsets corresponding to the partition $(1, 1, 1)$, which is the 2–tuple $(0, 0)$ in the labeling used here. This is parameterised by the open 6–ball $\operatorname{int}(I^3 \times \tilde{\Delta}_3)$, showing that dimensions can jump if the ordering on the boundary does not agree with that coming from the interior.

3 The two-sphere

3.1 Introduction

We now use the lexicographic cell structures developed in the previous section to study the finite subset spaces of the two-sphere. We use the standard cell structure for S^2 with a single two-cell, and to avoid clutter we will simply write S and $\tilde{S}$ for τ^S and $\tilde{\tau}^S$ respectively. Where the partition is written out explicitly we will use round brackets for τ^S and square brackets for $\tilde{\tau}^S$.

We begin in section 3.2 by describing the chain complexes of $\exp_k(S^2, *)$ and $\exp_k S^2$ and justifying the assertion of section 2.3 that ∂^λ and ∂^ν are commuting boundary operators. We look at the finite subset spaces of S^2 for several small values of k in section 3.3, and then prove Theorems 1 and 2 in section 3.4. The arguments bear a striking resemblance to those used to calculate the homology of $\exp_k \Gamma_n$ in [13]. The based space $\exp_k(S^2, *)$ again plays a prominent role, being easier to understand and having cleaner results, and we make use of the exact sequences of the pairs $(\exp_k(S^2, *), \exp_{k-1}(S^2, *))$ and $(\exp_k S^2, \exp_k(S^2, *))$. The main step is the calculation of $H_*(\exp_k(S^2, *) / \exp_{k-1}(S^2, *); \mathbf{Q})$. This involves the study of a combinatorially defined finite complex which may be recognised as the $(k-2)$–cube complex of [13] with some signs changed and certain edges "clipped".

3.2 The chain complexes

By Theorem 6 $\exp_k(S^2, *)$ has a cell structure with a single vertex in which the remaining cells are in one-to-one correspondence with the ordered partitions of the positive integers $j \leq k - 1$, and $\exp_k S^2$ has a cell structure obtained by adding additional cells corresponding to the ordered partitions of $j \leq k$. Let $\mathcal{S}_{\ell,j}$ and $\tilde{\mathcal{S}}_{\ell,j}$ be two copies of the free abelian group of rank $\binom{j-1}{\ell-1}$ generated by the ordered partitions of j of length ℓ. Each generator corresponds to a cell of dimension $j + \ell$ and we note that ∂^ν and ∂^λ map $\mathcal{S}_{\ell,j}$, $\tilde{\mathcal{S}}_{\ell,j}$ to $\mathcal{S}_{\ell,j-1}$, $\mathcal{S}_{\ell,j-1} \oplus \tilde{\mathcal{S}}_{\ell,j-1}$ and $\mathcal{S}_{\ell-1,j}$, $\mathcal{S}_{\ell-1,j} \oplus \tilde{\mathcal{S}}_{\ell-1,j}$ respectively. Since the boundary map is given by $\partial = \partial^\nu + (-1)^j \partial^\lambda$ it follows that the cellular chain groups of $\exp_k(S^2, *)$ and $\exp_k S^2$ may be arranged into double complexes with cell counts as shown in figure 4, and that $\partial^\nu, \partial^\lambda$ are commuting boundary operators as claimed. Moreover, we see immediately that for $k \geq 2$ we have $\chi(\exp_k(S^2, *)) = 2$ and $\chi(\exp_k S^2) = 3$.

In working with these complexes we will focus mainly on the chain complexes $(\mathcal{S}_{*,j}, \partial^\lambda)$ consisting of a single row, which amounts to working with the quotient
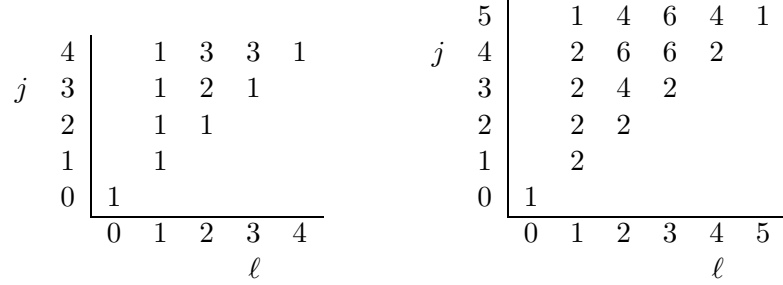
$$
\begin{array}{c|ccccc}
5 & 1 & 4 & 6 & 4 & 1 \\
j \quad 4 & 2 & 6 & 6 & 2 \\
3 & 2 & 4 & 2 \\
2 & 2 & 2 \\
1 & 2 \\
0 & 1 \\
\hline
& 0 & 1 & 2 & 3 & 4 & 5 \\
& & & & \ell
\end{array}
$$

$$
\begin{array}{c|cccc}
4 & 1 & 3 & 3 & 1 \\
j \quad 3 & 1 & 2 & 1 \\
2 & 1 & 1 \\
1 & 1 \\
0 & 1 \\
\hline
& 0 & 1 & 2 & 3 & 4 \\
& & & \ell
\end{array}
$$

Figure 4: Cell counts for $\exp_5(S^2, *)$ and $\exp_5 S^2$. Left: Cells of $\exp_k(S^2, *)$ may be arranged into a double complex with $\binom{j-1}{\ell-1}$ cells in position (ℓ, j), $1 \le \ell \le j \le k-1$, and one in position $(0,0)$. Since the alternating sum along a row of Pascal's triangle is zero we see immediately that $\chi(\exp_k(S^2, *)) = 2$ for $k \ge 2$. Right: Adding cells to get $\exp_k S^2$ doubles all rows other than the bottom one and adds an extra row of Pascal's triangle on top, giving $\chi(\exp_k S^2) = 3$ for $k \ge 2$.

spaces $\exp_{j+1}(S^2, *)/\exp_j(S^2, *)$. The (-1)–binomial co-efficients appearing in ∂^λ will be extraneous to our purpose so for clarity we will use the simplified boundary map (2.5), which we recall comes from using bases for $\mathcal{S}_{\ell,j}$ obtained by individually scaling each cell. Abusing notation we write (2.5) as

$$
\partial_{\mathbf{Q}}^\lambda S = \sum_{a=1}^{\ell} \frac{1 + (-1)^{s_a s_{a+1}}}{2} (-1)^{(a-1)} \mu_a(S).
$$

By analogy with the cube complex of section 2.3 of [13] we call the complex $(\mathcal{S}_{*,j}, \partial_{\mathbf{Q}}^\lambda)$ the "clipped cube complex", as the lattice of partitions of j, graded by length and partially ordered by refinement, forms a $(j-1)$–dimensional cube, and the boundary of S is again a linear combination of its neighbours of smaller degree. The adjective comes from the fact that $\partial_{\mathbf{Q}}^\lambda S$ omits the partitions obtained from S by amalgamating adjacent parts of odd size, so that these edges may be regarded as removed or "clipped". This is illustrated in figure 5, which should be compared with figure 2 of [13, p. 885]. We note that ∂^λ and $\partial_{\mathbf{Q}}^\lambda$ co-incide if each entry of S and $\mu_a(S)$ have size at most three, which holds for the top two chain groups of the complex. We will use this fact to calculate the top integral homology groups of $\exp_k(S^2, *)$ and $\exp_k S^2$. The distinction between ∂^λ and $\partial_{\mathbf{Q}}^\lambda$ occurs for the first time in the three dimensional clipped cube complex and can be seen in figure 5.

An understanding of the clipped cube complex allows us to calculate the rational homology of $\exp_k(S^2, *)$ inductively, using the long exact sequence of the pair $(\exp_k(S^2, *), \exp_{k-1}(S^2, *))$, and we pass to an understanding of $\exp_k S^2$ using
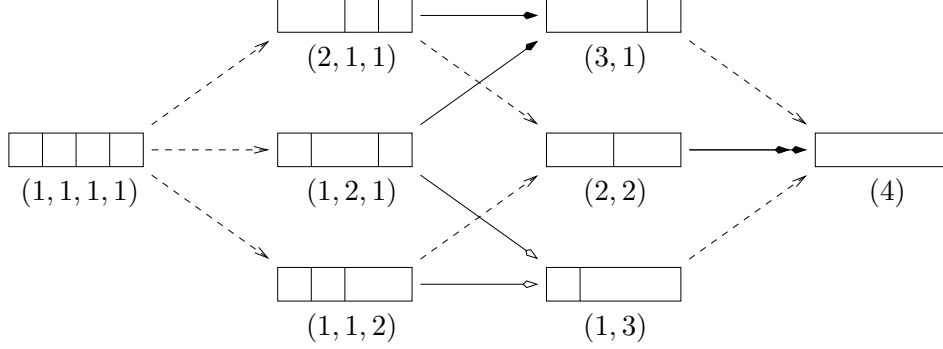
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Figure 5: The clipped cube complex; compare with figure 2 of [13], regarding the vertical dividing bars as the set $\{1, 2, 3\}$. The lattice of partitions of 4 forms a 3–dimensional cube, and the boundary of S is a signed sum of its neighbours with fewer parts, omitting those obtained by amalgamating adjacent parts of odd size. Positive terms are indicated by solid arrowheads and negative terms by empty arrowheads, and the omitted terms (the "clipped" edges) are shown by dotted arrows. The double-headed arrow from $(2, 2)$ to (4) shows where $\partial_{\mathbf{Q}}^{\lambda}$ differs from ∂^{λ}, as $\partial^{\lambda}(2, 2) = 2(4)$ but $\partial_{\mathbf{Q}}^{\lambda}(2, 2) = (4)$. We see that the homology of the clipped cube complex is 0 in dimensions 1 and 2 and $\mathbf{Z}$ in dimensions 3 and 4, generated by $(2, 1, 1) - (1, 2, 1) + (1, 1, 2)$ and $(1, 1, 1, 1)$ respectively, and that $\exp_5(S^2, *)/\exp_4(S^2, *)$ has an additional $\mathbf{Z}/2\mathbf{Z}$ summand generated by (4).

the long exact sequence of the pair $(\exp_k S^2, \exp_k(S^2, *))$. Our tool in doing so is the observation that the natural correspondence $\tilde{S} \leftrightarrow S$ between the tilded cells of $\exp_k S^2$ and the cells of $\exp_{k+1}(S^2, *)$ has the following algebraic consequence:

Lemma 1 *The spaces $\exp_{k+1}(S^2, *)$ and $\exp_k S^2/\exp_k(S^2, *)$ have the same homology.*

Proof Comparing $\exp_{k+1}(S^2, *)$ with the quotient space $\exp_k S^2/\exp_k(S^2, *)$ we see that one has a cell structure consisting of a single vertex, cells S satisfying $|S| \le k$, and boundary map

$$\partial S = \partial^{\nu} S + (-1)^{|S|} \partial^{\lambda} S,$$

and the second has a cell structure consisting of a single vertex, cells $\tilde{S}$ satisfying $|\tilde{S}| \le k$, and boundary map

$$\partial \tilde{S} = -\widetilde{\partial^{\nu} S} + (-1)^{|S|} \widetilde{\partial^{\lambda} S}.$$

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To account for the minus sign in the first term of the second map we twist the natural correspondence $\tilde{S} \leftrightarrow S$ by $(-1)^{|S|}$, obtaining an (algebraically defined) map $S \mapsto (-1)^{|S|}\tilde{S}$ inducing an isomorphism of chain complexes. $\qquad \square$

We emphasise the fact that we have proved this result algebraically, not topologically. The set map

$$\exp_{k+1}(S^2, *) \setminus \{\{*\}\} \rightarrow \exp_k S^2 / \exp_k(S^2, *) : \Lambda \mapsto \Lambda \setminus \{*\}$$

is discontinuous at each Λ of size k or less, and the map

$$\exp_k S^2 \rightarrow \exp_{k+1}(S^2, *) : \Lambda \mapsto \Lambda \cup \{*\}$$

does not descend to a continuous map on the quotient $\exp_k S^2 / \exp_k(S^2, *)$.

3.3 Small values of k

In this section we calculate the homology of $\exp_k(S^2, *)$ and $\exp_k S^2$ for several small values of k. Our aim is both to build familiarity with the chain complexes and to show by example that Theorem 2 does not give the full story on the integral homology of $\exp_k S^2$.

Since $\exp_1 S^2$ and $\exp_2(S^2, *)$ are both simply S^2 with its standard cell structure we begin with $\exp_2 S^2$ and $\exp_3(S^2, *)$. The lexicographic cell structure for $\exp_2 S^2$ has one cell $[1, 1]$ in dimension four, one cell $[2]$ in dimension three, two cells $[1]$ and (1) in dimension two, no cells in dimension one and a single vertex. The chain complex is

$$0 \longrightarrow \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{\left[\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right]} \mathbf{Z} \oplus \mathbf{Z} \longrightarrow 0 \longrightarrow \mathbf{Z} \longrightarrow 0,$$

and the homology is clearly

$$H_i(\exp_2 S^2) = \begin{cases} \mathbf{Z} & i = 0, 2, 4, \\ 0 & \text{else.} \end{cases}$$

Mapping $\exp_2 S^2$ to $\exp_3(S^2, *)$ by adding the vertex $*$ to each set rounds the brackets on the cells $[1, 1]$, $[2]$ and $[1]$, giving the chain complex

$$0 \longrightarrow \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{-1} \mathbf{Z} \longrightarrow 0 \longrightarrow \mathbf{Z} \longrightarrow 0$$

with homology

$$H_i(\exp_3(S^2, *)) = \begin{cases} \mathbf{Z} & i = 0, 4, \\ 0 & \text{else.} \end{cases} \tag{3.1}$$

Geometrically, $\exp_2 S^2 = \mathrm{Sym}^2 S^2 = \mathrm{Sym}^2 \mathbf{C}P^1$ may be identified with $\mathbf{C}P^2$ via the map

$$\{[z_0, w_0], [z_1, w_1]\} \mapsto [z_0 z_1, -(z_0 w_1 + z_1 w_0), w_0 w_1],$$

where the co-ordinates of the right-hand side are the co-efficients $[a, b, c]$ of the homogeneous quadratic $(z_0 W - w_0 Z)(z_1 W - w_1 Z)$ vanishing at $[z_0, w_0]$ and $[z_1, w_1]$. In this picture the quotient map $\exp_2 S^2 \xrightarrow{\cup\{*\}} \exp_3(S^2, *)$ induces the equivalence relation $[z^2, -2zw, w^2] \sim [z, -w, 0]$, identifying the degree two curve $b^2 - 4ac = 0$ with the degree one curve $a + b = 0$. As seen in (3.1) this kills the second homology and the resulting space $\exp_3(S^2, *)$ is homotopy equivalent to S^4.

To calculate the homology of $\exp_4(S^2, *)$ and $\exp_3 S^2$ we start with the clipped cube complex

$$
\begin{array}{ccc}
(1,1,1) & \xrightarrow{\ 0\ } & (2,1) \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 1} \\
(1,2) & \xrightarrow{\ 1\ } & (3)
\end{array}
\quad .
$$

This has homology groups $\mathbf{Z}$ in the top and middle dimensions, generated by $(1,1,1)$ and $(2,1) - (1,2)$ respectively, so

$$\tilde{H}_i(\exp_4(S^2, *)/\exp_3(S^2, *)) = \begin{cases} \mathbf{Z} & i = 5, 6, \\ 0 & \text{else.} \end{cases}$$

The long exact sequence of the pair $(\exp_4(S^2, *), \exp_3(S^2, *))$ then gives

$$0 \longrightarrow \tilde{H}_6(\exp_4(S^2, *)) \longrightarrow \tilde{H}_6(\exp_4(S^2, *)/\exp_3(S^2, *)) \longrightarrow 0$$

and

$$0 \longrightarrow \tilde{H}_5(\exp_4(S^2, *)) \longrightarrow \mathbf{Z} \xrightarrow{\ \partial\ } \mathbf{Z} \longrightarrow \tilde{H}_4(\exp_4(S^2, *)) \longrightarrow 0, \qquad (3.2)$$

the remaining segments having the form $0 \longrightarrow \tilde{H}_i(\exp_4(S^2, *)) \longrightarrow 0$. The boundary map $\partial \colon \tilde{H}_5(\exp_4(S^2, *)/\exp_3(S^2, *)) \to \tilde{H}_4(\exp_3(S^2, *))$ is ∂^ν which sends $(2,1) - (1,2)$ to $-2(1,1)$, so the middle map in (3.2) is multiplication by -2. Hence

$$\tilde{H}_i(\exp_4(S^2, *)) = \begin{cases} \mathbf{Z} & i = 6, \\ \mathbf{Z}/2\mathbf{Z} & i = 4, \\ 0 & \text{else,} \end{cases}$$

in which the nontrivial groups are generated by $(1,1,1)$ and $(1,1)$ respectively.

We now use Lemma 1 and the exact sequence of the pair $(\exp_3 S^2, \exp_3(S^2, *))$ to calculate the homology of $\exp_3 S^2$. The segments where $\tilde{H}_i(\exp_3 S^2)$ is not bracketed by zeroes are

$$0 \longrightarrow \tilde{H}_6(\exp_3 S^2) \longrightarrow \tilde{H}_6(\exp_3 S^2 / \exp_3(S^2, *)) \longrightarrow 0,$$

giving $\tilde{H}_6(\exp_3 S^2) \cong \mathbf{Z}$ generated by $[1, 1, 1]$, and

$$0 \longrightarrow \mathbf{Z} \longrightarrow \tilde{H}_4(\exp_3 S^2) \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow 0.$$

To see that this second sequence splits consider the cycle $[1, 1] - 2(1, 1)$. Under the right arrow this maps to the generator $[1, 1]$ of $\tilde{H}_4(\exp_3 S^2 / \exp_3(S^2, *))$, and the fact that the boundary of $[2, 1] - [1, 2]$ is $2([1, 1] - 2(1, 1))$ shows it has order two. Hence

$$\tilde{H}_i(\exp_3 S^2) = \begin{cases} \mathbf{Z} & i = 6, \\ \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} & i = 4, \\ 0 & \text{else,} \end{cases}$$

with generators $[1, 1, 1]$ in dimension six and $[1, 1]$ and $(1, 1)$ in dimension four. Using the universal co-efficient theorem we see that this result agrees with the $g = 0$ case of Theorem 5.

From figure 5 the homology of $\exp_5(S^2, *) / \exp_4(S^2, *)$ is $\mathbf{Z}$ in dimensions seven and eight and $\mathbf{Z}/2\mathbf{Z}$ in dimension five. The long exact sequence of $(\exp_5(S^2, *), \exp_4(S^2, *))$ has two nontrivial segments

$$0 \longrightarrow \tilde{H}_8(\exp_5(S^2, *)) \longrightarrow \tilde{H}_8(\exp_5(S^2, *) / \exp_4(S^2, *)) \longrightarrow 0$$

and

$$0 \longrightarrow \tilde{H}_7(\exp_5(S^2, *)) \longrightarrow \mathbf{Z} \overset{\partial}{\longrightarrow} \mathbf{Z} \longrightarrow \tilde{H}_6(\exp_5(S^2, *)) \longrightarrow 0$$

as above, and an additional nontrivial segment

$$0 \longrightarrow \tilde{H}_5(\exp_5(S^2, *)) \longrightarrow \mathbf{Z}/2\mathbf{Z} \overset{\partial}{\longrightarrow} \mathbf{Z}/2\mathbf{Z} \longrightarrow \tilde{H}_4(\exp_5(S^2, *)) \longrightarrow 0.$$

The first boundary map sends the generator $(2, 1, 1) - (1, 2, 1) + (1, 1, 2)$ to $-3(1, 1, 1)$, inducing multiplication by -3, and the second sends the generator (4) to $-(3)$. Since $\partial(2, 1) = -(1, 1) - (3)$ the second map is an isomorphism and

$$\tilde{H}_i(\exp_5(S^2, *)) = \begin{cases} \mathbf{Z} & i = 8, \\ \mathbf{Z}/3\mathbf{Z} & i = 6, \\ 0 & \text{else.} \end{cases}$$

Again using Lemma 1 and the exact sequence of $(\exp_4 S^2, \exp_4(S^2, *))$ we get

$$0 \longrightarrow \tilde{H}_8(\exp_4 S^2) \longrightarrow \tilde{H}_8(\exp_4 S^2 / \exp_4(S^2, *)) \longrightarrow 0,$$
$$0 \longrightarrow \mathbf{Z} \longrightarrow \tilde{H}_6(\exp_4 S^2) \longrightarrow \mathbf{Z}/3\mathbf{Z} \longrightarrow 0, \tag{3.3}$$

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$$
\begin{array}{cccc}
& & (3,1,1) & \\
& (2,1,1,1) & (1,3,1) & (4,1) \\
& (1,2,1,1) & (1,1,3) & (1,4) \\
(1,1,1,1,1) & (1,1,2,1) & (2,2,1) & (3,2) \\
& (1,1,1,2) & (1,2,2) & (2,3) \\
& & (2,1,2) &
\end{array}
\tag{5}
$$

Figure 6: Generators of the four dimensional clipped cube complex.

and

$$0 \longrightarrow \mathbf{Z}/2\mathbf{Z} \longrightarrow \tilde{H}_4(\exp_4 S^2) \longrightarrow 0.$$

The 6–cycle $[1,1,1] - 2(1,1,1)$ maps to the generator of the $\mathbf{Z}/3\mathbf{Z}$ term and three times it is the boundary of $[2,1,1] - [1,2,1] + [1,1,2]$, so the second sequence splits and

$$
\tilde{H}_i(\exp_4 S^2) =
\begin{cases}
\mathbf{Z} & i = 8, \\
\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z} & i = 6, \\
\mathbf{Z}/2\mathbf{Z} & i = 4, \\
0 & \text{else.}
\end{cases}
$$

We calculate one more example to show that the torsion becomes increasingly complicated as k increases. The example will require understanding the four dimensional clipped cube complex, shown in figure 6, and will further illustrate the ideas used in the general case.

As always, the boundary of a generator with all entries odd is zero, so we may consider the restriction of the boundary ∂^λ to the span of the generators with one or more even entries. Moreover, the boundaries of $(2,1,1,1)$ and its permutations lie in the span of $(3,1,1)$ and its permutations, so we may regard ∂_4^λ as a map to this subspace. With these conventions and the bases ordered as shown the matrices of the boundary maps are

$$
\partial_4^\lambda =
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & -1 & -1 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}, \qquad
\partial_3^\lambda =
\begin{bmatrix}
2 & 0 & 0 \\
0 & -2 & 0 \\
0 & 1 & 1 \\
-1 & 0 & -1
\end{bmatrix}, \qquad
\partial_2^\lambda =
\begin{bmatrix}
1 \\
1 \\
2 \\
2
\end{bmatrix}.
$$

Clearly, both ∂_4^λ and ∂_2^λ are surjective and ∂_3^λ is injective. The kernel of ∂_2^λ is equal to $\{2x+2y+z+w=0\}$ and by choosing a suitable basis for this subspace it can be seen that the image of ∂_3^λ has index two. In the now familiar pattern the long exact sequence of the pair $(\exp_6(S^2,*),\exp_5(S^2,*))$ gives

$$0 \longrightarrow \tilde{H}_{10}(\exp_6(S^2,*)) \longrightarrow \tilde{H}_{10}(\exp_6(S^2,*)/\exp_5(S^2,*)) \longrightarrow 0,$$

$$0 \longrightarrow \tilde{H}_9(\exp_6(S^2,*)) \longrightarrow \mathbf{Z} \stackrel{\partial}{\longrightarrow} \mathbf{Z} \longrightarrow \tilde{H}_8(\exp_6(S^2,*)) \longrightarrow 0,$$

and an additional segment

$$0 \longrightarrow \tilde{H}_7(\exp_6(S^2,*)) \longrightarrow \mathbf{Z}/2\mathbf{Z} \stackrel{\partial}{\longrightarrow} \mathbf{Z}/3\mathbf{Z} \longrightarrow \tilde{H}_6(\exp_6(S^2,*)) \longrightarrow 0.$$

The boundary map in the third sequence is necessarily zero, and the boundary map in the second sequence takes the generator $(2,1,1,1) - (1,2,1,1) + (1,1,2,1) - (1,1,1,2)$ to $-4(1,1,1,1)$. Hence

$$\tilde{H}_i(\exp_6(S^2,*)) = \begin{cases} \mathbf{Z} & i = 10, \\ \mathbf{Z}/4\mathbf{Z} & i = 8, \\ \mathbf{Z}/2\mathbf{Z} & i = 7, \\ \mathbf{Z}/3\mathbf{Z} & i = 6, \\ 0 & \text{else.} \end{cases}$$

Finally, from the long exact sequence of the pair $(\exp_5 S^2, \exp_5(S^2,*))$ we extract $\tilde{H}_{10}(\exp_5 S^2) \cong \mathbf{Z}$, $\tilde{H}_7(\exp_5 S^2) \cong \mathbf{Z}/2\mathbf{Z}$, and two short exact sequences

$$0 \longrightarrow \mathbf{Z} \longrightarrow \tilde{H}_8(\exp_5 S^2) \longrightarrow \mathbf{Z}/4\mathbf{Z} \longrightarrow 0,$$

$$0 \longrightarrow \mathbf{Z}/3\mathbf{Z} \longrightarrow \tilde{H}_6(\exp_5 S^2) \longrightarrow \mathbf{Z}/3\mathbf{Z} \longrightarrow 0.$$

As usual the first is split by $[1,1,1,1] - 2(1,1,1,1)$, and as in (3.3) the second is split by $[1,1,1] - 2(1,1,1)$. Hence

$$\tilde{H}_i(\exp_5 S^2) = \begin{cases} \mathbf{Z} & i = 10, \\ \mathbf{Z} \oplus \mathbf{Z}/4\mathbf{Z} & i = 8, \\ \mathbf{Z}/2\mathbf{Z} & i = 7, \\ \mathbf{Z}/3\mathbf{Z} \oplus \mathbf{Z}/3\mathbf{Z} & i = 6, \\ 0 & \text{else.} \end{cases}$$

3.4 The general case

We now turn to the general case. As we have seen, the integral homology of $\exp_k S^2$ becomes increasingly complicated, with more and more torsion arising

as k increases. We therefore give a full answer only with respect to rational coefficients, where the situation is much cleaner, and limit ourselves to calculating only the top three integral groups. Not only are these groups computationally tractable but they also exhibit great regularity.

We begin by calculating the homology of the clipped cube complex. We will be working mainly with the top end, where the partitions consist mostly of ones, and for compactness of notation we write $\mathbf{1}(k)$ for the partition $(1, \ldots, 1)$ of k as a sum of k ones, or just $\mathbf{1}$ if k is understood. We write $\mathbf{2}_i(k)$ for the partition $(1, \ldots, 2, \ldots, 1)$ with a single 2 occurring in the ith place, and write $\mathbf{3}_i$, $\mathbf{4}_i$ for the analogous partition with a single 3 or 4. By extension, we write $\mathbf{2}_i\mathbf{3}_j$ for the partition with a 2 in the ith place, a 3 in the jth place, and the remaining entries ones. With these conventions, the top four chain groups of the clipped cube complex may be written

$$
\begin{aligned}
\mathcal{S}_{k,k} &= \mathrm{span}\{\mathbf{1}\}, \\
\mathcal{S}_{k-1,k} &= \mathrm{span}\{\mathbf{2}_i | 1 \leq i \leq k-1\}, \\
\mathcal{S}_{k-2,k} &= \mathrm{span}\{\mathbf{3}_i | 1 \leq i \leq k-2\} \oplus \mathrm{span}\{\mathbf{2}_i\mathbf{2}_j | 1 \leq i < j \leq k-2\}, \\
\mathcal{S}_{k-3,k} &= \mathrm{span}\{\mathbf{4}_i | 1 \leq i \leq k-3\} \oplus \mathrm{span}\{\mathbf{2}_i\mathbf{3}_j | i \neq j, 1 \leq i,j \leq k-3\} \\
&\quad \oplus \mathrm{span}\{\mathbf{2}_i\mathbf{2}_j\mathbf{2}_m | 1 \leq i < j < m \leq k-3\}.
\end{aligned}
$$

Fortunately, there will be no need to count beyond four or work with juxtapositions of more than two characters.

Since the boundary of $\mathbf{1}$ is zero we see immediately that the kth homology of the clipped cube complex is $\mathbf{Z}$. The group H_{k-1} is calculated almost as easily: the boundary of $\mathbf{2}_i$ is given by

$$
\partial \mathbf{2}_i = \begin{cases} \mathbf{3}_1 & i = 1, \\ (-1)^i(\mathbf{3}_{i-1} - \mathbf{3}_i) & 2 \leq i \leq k-2, \\ \mathbf{3}_{k-2} & i = k-1, \end{cases}
$$

and taking alternating sums gives

$$
\partial \sum_{i=1}^{j}(-1)^{i-1}\mathbf{2}_i = \begin{cases} \mathbf{3}_j & 1 \leq j \leq k-2, \\ 0 & j = k-1. \end{cases} \tag{3.4}
$$

This shows that H_{k-1} is isomorphic to $\mathbf{Z}$ also, with generator $\sum_{i=1}^{k-1}(-1)^{i-1}\mathbf{2}_i$.

The remaining homology groups are rather more difficult to calculate and are the subject of the following lemma:

Lemma 2 *The clipped cube complex* $(\mathcal{S}_{*,k}, \partial_{\mathbf{Q}}^{\lambda})$ *is exact over* $\mathbf{Q}$ *at each* $\mathcal{S}_{\ell,k}$, $1 \leq \ell \leq k-2$, *and exact over* $\mathbf{Z}$ *at* $\mathcal{S}_{k-2,k}$.

Proof The proof is in two parts: we first prove exactness over $\mathbf{Z}$ at $\mathcal{S}_{k-2,k}$ directly, and then prove exactness at the remaining groups by induction over k, using the result for the clipped cube complexes $(\mathcal{S}_{*,k-1}, \partial_{\mathbf{Q}}^{\lambda})$ and $(\mathcal{S}_{*,k-2}, \partial_{\mathbf{Q}}^{\lambda})$. The direct proof of exactness at $\mathcal{S}_{k-2,k}$ is also required to complete the induction step. For simplicity, in what follows we write ∂ for $\partial_{\mathbf{Q}}^{\lambda}$.

By (3.4) the image of ∂_{k-1} is the span of $\{3_i | 1 \leq i \leq k-2\}$, so to prove exactness at $\mathcal{S}_{k-2,k}$ over $\mathbf{Z}$ it suffices to show that ∂_{k-2} restricted to the span of $\{2_i 2_j | 1 \leq i < j \leq k-2\}$ is injective. Let U be this span and write $\mathcal{S}_{k-3,k} = V \oplus W$, where

$$V = \operatorname{span}\{4_i | 1 \leq i \leq k-3\} \oplus \operatorname{span}\{3_i 2_j | 1 \leq i < j \leq k-3\},$$
$$W = \operatorname{span}\{2_i 3_j | 1 \leq i < j \leq k-3\} \oplus \operatorname{span}\{2_i 2_j 2_m | 1 \leq i < j < m \leq k-3\}.$$

Let $\bar{V} = (V \oplus W)/W$, and observe that $\operatorname{rank} U = \operatorname{rank} \bar{V} = \binom{k-2}{2}$. The boundary induces a map $\bar{\partial} \colon U \to \bar{V}$, and we claim that with respect to suitable orderings of the bases $\{2_i 2_j | i < j\}$ and $\{4_i\} \cup \{3_i 2_j | i < j\}$ the matrix of $\bar{\partial}$ is upper triangular.

To see this, lexicographically order each basis according to $(j - i, i)$, where for this purpose we regard 4_i as $3_i 2_i$ and place it in the order as $(0, i)$. The boundary of $2_i 2_j$ is given by

$$\partial 2_i 2_j = \begin{cases} (-1)^i (3_{i-1} 2_i - 4_i + 2_i 3_{i+1}) & j = i+1, \\ (-1)^i (3_{i-1} 2_{j-1} - 3_i 2_{j-1}) + (-1)^j (2_i 3_{j-1} - 2_i 3_j) & j-1 \geq 2, \end{cases}$$

in which the terms containing 3_{i-1} should be omitted if $i = 1$, and the terms containing 3_j should be omitted if $j = k-2$. Passing to the quotient $\bar{V}$ we have

$$\bar{\partial} 2_i 2_j = \begin{cases} (-1)^i (3_{i-1} 2_i - 4_i) & j = i+1, \\ (-1)^i (3_{i-1} 2_{j-1} - 3_i 2_{j-1}) & j-1 \geq 2, \end{cases}$$

and in each case we see that the least element occurring in $\bar{\partial} 2_i 2_j$ is the one ordered by $(j-i-1, i)$. This implies that the matrix of $\bar{\partial}$ is lower triangular with each diagonal entry equal to ± 1, and we conclude that $\bar{\partial}$ is an isomorphism.

We now show by induction on k that the boundary map has rank $\binom{k-2}{\ell-2}$ at $\mathcal{S}_{\ell,k}$ for each $1 \leq \ell \leq k-2$. This is easily verified for the cases $k = 3$ and $k = 4$ considered in section 3.3 and we use these as the base for the induction. The case $\ell = k-2$ is the one just proved and is required for the inductive step. For

$1 \leq \ell \leq k - 3$ we relate $\mathcal{S}_{\ell,k}$ to $\mathcal{S}_{*,k-1}$ and $\mathcal{S}_{*,k-2}$ by examining the size of the last element of each partition. Let

$$\mathcal{S}_{\ell,k}^i = \operatorname{span}\{S \in \mathcal{S}_{\ell,k} | s_\ell = i\}$$

for $i = 1, 2$,

$$\mathcal{S}_{\ell,k}^3 = \operatorname{span}\{S \in \mathcal{S}_{\ell,k} | s_\ell \geq 3\},$$

and consider the "append" and "plus" operators

$$A_i(S) = (s_1, \ldots, s_\ell, i)$$

for $i = 1, 2$, and

$$P_2(S) = (s_1, \ldots, s_\ell + 2).$$

We have

$$A_i \colon \mathcal{S}_{\ell-1,k-i} \xrightarrow{\cong} \mathcal{S}_{\ell,k}^i,$$

$$P_2 \colon \mathcal{S}_{\ell,k-2} \xrightarrow{\cong} \mathcal{S}_{\ell,k}^3,$$

and moreover

$$\partial A_i S \in A_i \partial S + \mathcal{S}_{\ell,k}^3$$

for $i = 1, 2$, and

$$\partial P_2 S = P_2 \partial S \tag{3.5}$$

since P_2 does not change the parity of the last entry of S. We note also that (3.5) depends on the fact we are using the boundary map $\partial_{\mathbf{Q}}^\lambda$ instead of ∂^λ.

Since $\mathcal{S}_{\ell,k} = \mathcal{S}_{\ell,k}^1 \oplus \mathcal{S}_{\ell,k}^2 \oplus \mathcal{S}_{\ell,k}^3$ it follows that the matrix of ∂_ℓ can be written in the block form

$$D_{\ell,k} = \begin{bmatrix} D_{\ell-1,k-1} & 0 & 0 \\ 0 & D_{\ell-1,k-2} & 0 \\ E & F & D_{\ell,k-2} \end{bmatrix},$$

in which $D_{i,j}$ is the matrix of ∂ acting on $\mathcal{S}_{i,j}$. Hence

$$\operatorname{rank} \partial_\ell = \operatorname{rank} D_{\ell-1,k-1} + \operatorname{rank} D_{\ell-1,k-2} + \operatorname{rank} D_{\ell,k-2}$$
$$= \binom{k-3}{\ell-3} + \binom{k-4}{\ell-3} + \binom{k-4}{\ell-2}$$
$$= \binom{k-2}{\ell-2}$$

as desired.

We complete the proof using a rank argument. We have

$$\operatorname{rank} \ker \partial_\ell = \operatorname{rank} \mathcal{S}_{\ell,k} - \operatorname{rank} \partial_\ell$$
$$= \binom{k-1}{\ell-1} - \binom{k-2}{\ell-2}$$
$$= \binom{k-2}{\ell-1}$$
$$= \operatorname{rank} \partial_{\ell+1},$$

from which the result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Since $\partial_{\mathbf{Q}}^\lambda$ and ∂^λ co-incide on $\mathcal{S}_{k,k}$ and $\mathcal{S}_{k-1,k}$ we may restate this lemma and the calculations preceding it in the following form:

Lemma 3 *The rational homology of the space $\exp_k(S^2,*)/\exp_{k-1}(S^2,*)$ vanishes except in dimensions $2k-2$ and $2k-3$. With respect to integer co-efficients the top three groups are $\mathbf{Z}$ in dimension $2k-2$, generated by $1(k-1)$, $\mathbf{Z}$ in dimension $2k-3$, generated by $\displaystyle\sum_{i=1}^{k-2}(-1)^{i-1}2_i(k-1)$, and 0 in dimension $2k-4$.*

We are now in a position to prove Theorems 1 and 2. The argument should be very familiar from the cases considered in section 3.3.

Proof of Theorems 1 and 2 We first prove both results for $\exp_k(S^2,*)$, using induction on k. The top end of the long exact sequence of the pair $(\exp_k(S^2,*),\exp_{k-1}(S^2,*))$ gives

$$0 \longrightarrow \tilde{H}_{2k-2}(\exp_k(S^2,*)) \longrightarrow \tilde{H}_{2k-2}(\exp_k(S^2,*)/\exp_{k-1}(S^2,*)) \longrightarrow 0,$$

implying $\tilde{H}_{2k-2}(\exp_k(S^2,*)) \cong \mathbf{Z}$ with generator $1(k-1)$. The next segment is

$$0 \longrightarrow \tilde{H}_{2k-3}(\exp_k(S^2,*)) \longrightarrow \mathbf{Z} \xrightarrow{\partial} \mathbf{Z} \longrightarrow \tilde{H}_{2k-4}(\exp_k(S^2,*)) \longrightarrow 0,$$

where the boundary map in the middle, from $\tilde{H}_{2k-3}(\exp_k(S^2,*)/\exp_{k-1}(S^2,*))$ to $\tilde{H}_{2k-4}(\exp_{k-1}(S^2,*))$, is given by ∂^ν. Since $\partial^\nu 2_i(k-1) = (-1)^i 1(k-2)$ this sends the generator $\displaystyle\sum_{i=1}^{k-2}(-1)^{i-1}2_i(k-1)$ to $(2-k)(1(k-2))$, inducing multiplication by $2-k$. It follows that $\tilde{H}_{2k-3}(\exp_k(S^2,*))$ is zero and that $\tilde{H}_{2k-4}(\exp_k(S^2,*))$ is isomorphic to $\mathbf{Z}/(k-2)\mathbf{Z}$, generated by the top homology

class of $\exp_{k-1}(S^2, *)$. We show that the remaining groups vanish over $\mathbf{Q}$ using the induction hypothesis and the exact sequence

$$\tilde{H}_i(\exp_{k-1}(S^2, *)) \longrightarrow \tilde{H}_i(\exp_k(S^2, *)) \longrightarrow \tilde{H}_i(\exp_k(S^2, *)/\exp_{k-1}(S^2, *))$$

(rational co-efficients omitted). Since the outer groups are zero for $i \leq 2k - 5$ the middle group is too.

We now prove the results for $\exp_k S^2$, using the results just proved, Lemma 1, and the long exact sequence of the pair $(\exp_k S^2, \exp_k(S^2, *))$. At the top end we have

$$0 \to \tilde{H}_{2k}(\exp_k S^2) \to \tilde{H}_{2k}(\exp_k S^2/\exp_k(S^2, *)) \to 0 \to \tilde{H}_{2k-1}(\exp_k S^2) \to 0$$

giving $\tilde{H}_{2k}(\exp_k S^2) \cong \mathbf{Z}$ generated by $\tilde{1}(k)$ and $\tilde{H}_{2k-1}(\exp_k S^2) \cong \{0\}$. The next segment is

$$0 \longrightarrow \mathbf{Z} \longrightarrow \tilde{H}_{2k-2}(\exp_k S^2) \longrightarrow \mathbf{Z}/(k-1)\mathbf{Z} \longrightarrow 0$$

in which the subgroup $\mathbf{Z}$ is generated by $1(k-1)$ and the quotient $\mathbf{Z}/(k-1)\mathbf{Z}$ is generated by $\tilde{1}(k-1)$. This short exact sequence splits: the cycle $\tilde{1}(k-1) - 2(1(k-1))$ maps to $\tilde{1}(k-1)$ in the quotient, and

$$\partial \sum_{i=1}^{k-1} (-1)^{i-1} \tilde{2}_i(k) = (k-1)\big(\tilde{1}(k-1) - 2(1(k-1))\big),$$

showing that this class has order $k-1$. This completes the proof of Theorem 2, and the exact sequence

$$\tilde{H}_i(\exp_k(S^2, *); \mathbf{Q}) \longrightarrow \tilde{H}_i(\exp_k S^2; \mathbf{Q}) \longrightarrow \tilde{H}_i(\exp_k S^2/\exp_k(S^2, *); \mathbf{Q})$$

yields $0 \longrightarrow \tilde{H}_i(\exp_k S^2; \mathbf{Q}) \longrightarrow 0$ for $i \leq 2k - 3$, completing the proof of Theorem 1. $\qquad\square$

4   Higher genus surfaces

In this section we prove Theorem 3 on the top homology of the finite subset spaces of a closed surface, and Theorem 4 on the degree of a map $\exp_k f$ induced by a map $f \colon \Sigma \to \Sigma'$ between closed oriented surfaces.

We give Σ the standard cell structure consisting of a single vertex, $2g$ or g edges, and a single two-cell attached along the word $w_+ = [e_1, e_{1+g}] \cdots [e_g, e_{2g}]$ if Σ is orientable, $w_- = e_1^2 \cdots e_g^2$ if Σ is non-orientable. With respect to this

cell structure the $2k-$ and $(2k-1)$–cells of the lexicographic cell structure for $\exp_k \Sigma$ are the single cell $\tilde{1}(k)$ in dimension $2k$ and the cells

$$\left\{\tilde{2}_i(k)|1 \le i \le k-1\right\} \cup \left\{(\tilde{1}(k-1)) \cup \tilde{\sigma}_i^1|1 \le i \le \dim H_1(\Sigma; \mathbf{Z}/2\mathbf{Z})\right\}$$

in dimension $2k-1$. In order to calculate boundaries we need to know the chain maps $(\exp w_+)_\sharp$ and $(\exp w_-)_\sharp$. By the discussion at the end of section 3.3 of [13] it suffices to calculate the images of the generators $\tilde{\sigma}^1$ and $\tilde{\sigma}^2$ of the chain ring, and moreover these are given by

$$(\exp w_+)_\sharp \tilde{\sigma}^1 = 0, \qquad\qquad (\exp w_-)_\sharp \tilde{\sigma}^1 = 2\sum_{i=1}^{g} \tilde{\sigma}_i^1,$$

$$(\exp w_+)_\sharp \tilde{\sigma}^2 = 2\sum_{i=1}^{g} \tilde{\sigma}_i^1 \cup \tilde{\sigma}_{i+g}^1, \quad (\exp w_-)_\sharp \tilde{\sigma}^2 = 2\sum_{i=1}^{g} \tilde{\sigma}_i^2 + 4\sum_{i<j} \tilde{\sigma}_i^1 \cup \tilde{\sigma}_j^1.$$

This gives us all the ingredients we need and we now proceed to the proofs.

Proof of Theorem 3 Since $\partial^\nu \tilde{1} = \partial^\lambda \tilde{1} = 0$ the boundary of the cell $\tilde{1}(k)$ is equal to $\pm\partial^\gamma \tilde{1}(k)$. We have

$$\partial^\gamma \tilde{1}(k) = \begin{cases} 0 & \Sigma \text{ orientable}, \\ 2\sum_{i=1}^{g} \tilde{1}(k-1) \cup \tilde{\sigma}_i^1 & \Sigma \text{ non-orientable}, \end{cases} \qquad (4.1)$$

and we see immediately that

$$H_{2k}(\exp_k \Sigma) = \begin{cases} \mathbf{Z} & \text{if } \Sigma \text{ is orientable}, \\ 0 & \text{if } \Sigma \text{ is non-orientable}. \end{cases}$$

To calculate the kernel of ∂_{2k-1} we make the following observation. Let F_d be the span of the d–cells of the form τ^S, $\tilde{\tau}^S$, and let E_d be the span of the d–cells with a σ^J or $\tilde{\sigma}^J$ factor with $|J| > 0$. With respect to the decompositions of the cellular chain groups $\mathcal{C}_d$ as $F_d \oplus E_d$ the matrix of ∂ has the block form

$$D_d = \begin{bmatrix} A_d & 0 \\ B_d & C_d \end{bmatrix},$$

and if A_d is injective then $\ker \partial = \ker C_d$. Applying this with $d = 2k-1$ we have

$$F_{2k-1} = \text{span}\{\tilde{2}_i(k)|1 \le i \le k-1\},$$
$$E_{2k-1} = \text{span}\{\tilde{1}(k-1) \cup \tilde{\sigma}_i^1|1 \le i \le \dim H_1(\Sigma; \mathbf{Z}/2\mathbf{Z})\},$$

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and A_{2k-1} is injective since $A_{2k-1}\tilde{2}_i$ is equal to the boundary of this cell as a cell in $\exp_k S^2$. It follows that the $(2k-1)$–cycles are contained in the span of $\{\tilde{1}(k-1)\cup\tilde{\sigma}_i^1 | 1 \le i \le \dim H_1(\Sigma; \mathbf{Z}/2\mathbf{Z})\}$. By equation (4.1) we have

$$\partial\big(\tilde{1}(k-1)\cup\tilde{\sigma}_i^1\big) = \begin{cases} 0 & \Sigma \text{ orientable,} \\ \pm 2\displaystyle\sum_{j\neq i}\tilde{1}(k-2)\cup\tilde{\sigma}_j^1\cup\tilde{\sigma}_i^1 & \Sigma \text{ non-orientable.} \end{cases}$$

If Σ is non-orientable then

$$\partial\sum_{i=1}^{g} a_i\tilde{1}(k-1)\cup\tilde{\sigma}_i^1 = \pm 2\sum_{i<j}(a_i - a_j)\tilde{1}(k-2)\cup\tilde{\sigma}_i^1\cup\tilde{\sigma}_j^1,$$

implying

$$\ker\partial_{2k-1} = \begin{cases} \operatorname{span}\big\{\tilde{1}(k-1)\cup\tilde{\sigma}_i^1 | 1 \le i \le 2g\big\} & \Sigma \text{ orientable,} \\ \operatorname{span}\bigg\{\displaystyle\sum_{i=1}^{g}\tilde{1}(k-1)\cup\tilde{\sigma}_i^1\bigg\} & \Sigma \text{ non-orientable.} \end{cases}$$

It now follows that

$$H_{2k-1}(\exp_k\Sigma) = \begin{cases} \mathbf{Z}^{2g} & \text{if } \Sigma \text{ is orientable,} \\ \mathbf{Z}/2\mathbf{Z} & \text{if } \Sigma \text{ is non-orientable,} \end{cases}$$

completing the proof of Theorem 3. $\square$

Next we prove Theorem 4, which we recall states that

$$\deg\exp_k f = (\deg f)^k$$

if $f\colon \Sigma \to \Sigma'$ is a map between closed oriented surfaces. The result is an almost immediate consequence of the fact that $\exp_k\Sigma \setminus \exp_{k-1}\Sigma$ may be consistently oriented by (1.2). We caution that this orientation disagrees with the orientation on $\tilde{1}(k)$ for some k.

Proof of Theorem 4 Given a map $f\colon \Sigma \to \Sigma'$ let $p_1, \ldots, p_k$ be distinct points in Σ' and perturb f to be transverse to each p_i. Each point p_i then has finitely many preimages which we label $q_{i,1}, \ldots, q_{i,r_i}$. Let $\Lambda = \{p_1, \ldots, p_k\} \in \exp_k\Sigma'$. Under $\exp_k f$ the preimage of Λ consists of the $r_1 \cdots r_k$ points

$$\Lambda_{s_1\cdots s_k} = \{q_{1,s_i}, \ldots, q_{k,s_k}\},$$

and perturbing the characteristic maps of the 2–cells of Σ, Σ' if necessary we may assume that all points in question lie in the top dimensional cells $\tilde{1}(k)$.

With respect to the canonical orientation on $\exp_k\Sigma\setminus\exp_{k-1}\Sigma$ the sign of $\Lambda_{s_1\cdots s_k}$ is the product $\prod_i \operatorname{sign} q_{i,s_i}$, and we have immediately

$$\deg \exp_k f = \prod_{i=1}^{k}\sum_{j=1}^{r_i} \operatorname{sign} q_{i,j} = (\deg f)^k.$$

$\square$

5 A homotopy model for the third finite subset space

We now pursue our second direction in the study of the finite subset spaces of closed surfaces. We begin by constructing a homotopy model for the third finite subset space of a closed orientable surface, and then we use this in section 6 to calculate the cohomology of $\exp_3\Sigma$.

5.1 The model

To construct a homotopy model for $\exp_3\Sigma$ we begin with the intermediate quotient $\operatorname{Sym}^3\Sigma = \Sigma^3/S_3$. Inside $\operatorname{Sym}^3\Sigma$ we have the preimage of $\exp_2\Sigma$, namely the branch locus $\mathcal{D}$ consisting of the quotient of the diagonals of $\Sigma \times \Sigma \times \Sigma$. Instead of quotienting $\mathcal{D}$ further to get $\exp_3\Sigma$ we attach the mapping cylinder of $\mathcal{D} \to \exp_2\Sigma$ to obtain a homotopy equivalent space $E_3\Sigma$. Fortunately $\mathcal{D}$ is simply a copy of $\Sigma \times \Sigma$ and $\mathcal{D} \to \exp_2\Sigma$ is the quotient map $\Sigma \times \Sigma \to \operatorname{Sym}^2\Sigma$, so the construction is all in terms of known spaces and maps.

Concretely, let $q_k\colon \Sigma^k \to \operatorname{Sym}^k\Sigma$ be the quotient map and $\Delta\colon \Sigma \to \Sigma \times \Sigma$ the diagonal map. Then $\mathcal{D}$ is the image of $\Sigma \times \Sigma$ under the map $\iota = q_3 \circ \operatorname{id}\times\Delta$, and is homeomorphic to $\Sigma \times \Sigma$ since ι is injective, $\Sigma \times \Sigma$ is compact, and $\operatorname{Sym}^3\Sigma$ is Hausdorff. Further, $\exp_3\Sigma$ is obtained from $\operatorname{Sym}^3\Sigma$ by identifying $\iota(a,b) = q_3(a,b,b)$ with $\iota(b,a) = q_3(b,a,a)$. Let

$$M_{q_2} = \frac{(\Sigma \times \Sigma \times I) \amalg \operatorname{Sym}^2\Sigma}{(x,1) \sim q_2(x)}$$

be the mapping cylinder of q_2, and note that we adopt the convention that mapping cylinders are attached to the target at $1 \in I$ rather than 0. We obtain our model for $\exp_3\Sigma$ by attaching M_{q_2} to $\operatorname{Sym}^3\Sigma$ along $\mathcal{D}$ and $\Sigma \times \Sigma \times \{0\}$, namely

$$E_3\Sigma = \operatorname{Sym}^3\Sigma \cup_{\Sigma\times\Sigma} M_{q_2}$$
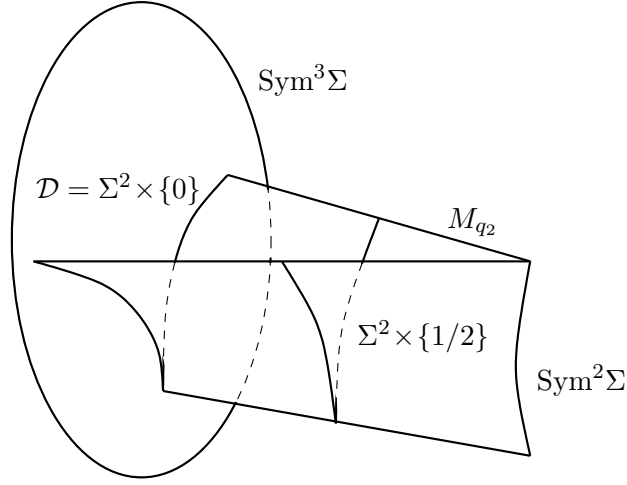$$= \frac{\operatorname{Sym}^3\Sigma \amalg M_{q_2}}{\iota(x) \sim (x,0)}.$$

Figure 7: A schematic picture of the space $E_3\Sigma$. As we shall see in section 5.2, $\mathrm{Sym}^3\Sigma$ is a complex 3–manifold with the branch locus $\mathcal{D} \cong \Sigma^2$ embedded with a cusp along the diagonal. Σ^2 maps two-to-one to $\mathrm{Sym}^2\Sigma$ via q_2, and we form $E_3\Sigma$ by attaching the mapping cylinder M_{q_2} to $\mathcal{D}$ along $\Sigma^2 \times \{0\}$.

The space $E_3\Sigma$ is shown schematically in figure 7. The subscript 3 reflects the hope that a similar construction may apply for $k \geq 4$, perhaps using several mapping cylinders to successively quotient $\mathrm{Sym}^k\Sigma$ to $\exp_k\Sigma$ in several stages. However, such a generalisation is complicated by the increasing complexity of the branch locus: for example, when $k = 4$ it is $\Sigma \times \mathrm{Sym}^2\Sigma$ with an embedded copy of $\Sigma \times \Sigma$ quotiented to $\mathrm{Sym}^2\Sigma$.

The cornerstone of the next section is the following lemma:

Lemma 4 *The spaces $E_3\Sigma$ and $\exp_3\Sigma$ are homotopy equivalent.*

We prove the lemma after a brief digression on known facts about symmetric products of surfaces. Not all of what follows is central to our argument, but it nonetheless serves to give a fuller picture of the construction.

5.2   Symmetric products of surfaces

We recall that the symmetric product of an orientable surface is a manifold, and moreover that a complex structure on Σ leads to a complex structure on $\mathrm{Sym}^k\Sigma$. Local co-ordinates about $\{p_1, \ldots, p_k\}$ are given by the elementary

symmetric functions in the local complex co-ordinates about the p_i, and are obtained by regarding the points as the zeroes of a polynomial (see Griffiths and Harris [4, p. 326]). In particular $\operatorname{Sym}^n \mathbf{C}P^1$ may be identified with the non-vanishing homogeneous polynomials of degree n in two variables, modulo scaling, and as such is equal to $\mathbf{C}P^n$.

A polynomial p has repeated roots if and only if its discriminant is zero. The discriminant is a polynomial in the co-efficients of p (see Lang [7, pp. 192–194]) and it follows that the branch locus, the image of the diagonals of Σ^k, is locally given by the vanishing of a polynomial and is therefore an algebraic variety. Specialising to $k = 3$, for a suitable choice of complex co-ordinates (a, b, c) the branch locus $\mathcal{D}$ is locally the set $b^2 = c^3$. We see that $\mathcal{D}$ has a cusp along $b = c = 0$, which is precisely the image of the main diagonal in $\Sigma \times \Sigma \times \Sigma$.

5.3   The proof of homotopy equivalence

Consider the mapping cylinder of $q\colon \operatorname{Sym}^3\Sigma \to \exp_3\Sigma$. This deformation retracts to $\exp_3\Sigma$ and contains $E_3\Sigma$ as a subspace, and our aim is to show that it also deformation retracts to $E_3\Sigma$. To do this it suffices to show that $\operatorname{Sym}^3\Sigma \times I$ deformation retracts to $\operatorname{Sym}^3\Sigma \times \{0\} \cup \mathcal{D} \times I$, as such a homotopy will descend setwise to the quotient M_q and be continuous there.

The existence of a deformation retraction from $\operatorname{Sym}^3\Sigma \times I$ to $\operatorname{Sym}^3\Sigma \times \{0\} \cup \mathcal{D} \times I$ follows from results in Bredon [1, pp. 431–432] and Dugundji [3, pp. 327–328] and the existence of a neighbourhood $U \supseteq \mathcal{D}$ that strongly deforms to $\mathcal{D}$ in $\operatorname{Sym}^3\Sigma$. We will prove it in this way, using the fact that $\operatorname{Sym}^3\Sigma$ and $\mathcal{D}$ are both compact manifolds to construct U. However, an approach of perhaps greater generality might be to realise $\mathcal{D}$ as a subcomplex of $\operatorname{Sym}^3\Sigma$ and appeal to Hatcher [6, Prop. 0.16]. Hatcher [6, pp. 482–483] gives a construction of an S_k- equivariant simplicial structure on X^k for simplicial X, and Lemma 4 would follow from checking whether this contains the diagonals as a subcomplex.

Although $\operatorname{Sym}^3\Sigma$ and $\mathcal{D}$ are both manifolds the existence of the desired neighbourhood U does not simply follow from the tubular neighbourhood theorem, since $\mathcal{D}$ is not smoothly embedded. We therefore resort to more hands-on means, and use the fact that manifolds are Euclidean neighbourhood retracts. Embed $M = \operatorname{Sym}^3\Sigma$ in some $\mathbf{R}^n$. Then there are neighbourhoods V of M, W of $\mathcal{D}$, and retractions $r_M\colon V \to M$, $r_W\colon W \to \mathcal{D}$. The neighbourhood W may be taken sufficiently small that the linear homotopy from $W \hookrightarrow \mathbf{R}^n$ to r_W remains in V, and we post-compose this with r_M and intersect W with M to get the desired neighbourhood and deformation.

6 The calculation of $H^*(\exp_3\Sigma)$

6.1 Introduction

To calculate the cohomology of $E_3\Sigma$ we use the Mayer-Vietoris sequence and the obvious decomposition

$$E_3\Sigma = (E_3\Sigma \setminus \mathrm{Sym}^2\Sigma) \cup (E_3\Sigma \setminus \mathrm{Sym}^3\Sigma).$$

The pieces are homotopy equivalent to $\mathrm{Sym}^3\Sigma$ and $\mathrm{Sym}^2\Sigma$ respectively, and intersect in $\Sigma \times \Sigma \times (0,1) \simeq \Sigma \times \Sigma$, leading to a long exact sequence

$$\cdots \to H^i(E_3\Sigma) \to H^i(\mathrm{Sym}^3\Sigma) \oplus H^i(\mathrm{Sym}^2\Sigma) \to H^i(\Sigma^2) \to H^{i+1}(E_3\Sigma) \to \cdots.$$

Before proceeding we describe the rings $H^*(\Sigma^k)$ and $H^*(\mathrm{Sym}^k\Sigma)$. Recall that integer co-efficients are to be assumed except where specified otherwise.

6.2 The rings $H^*(\Sigma^k)$ and $H^*(\mathrm{Sym}^k\Sigma)$

Let $\alpha_1, \ldots, \alpha_{2g}$ be generators for $H^1(\Sigma)$ such that

$$\alpha_i\alpha_j = \begin{cases} 0 & |i - j| \neq g, \\ \beta & j = i + g, \end{cases}$$

where β is a generator of $H^2(\Sigma)$. Since $H^*(\Sigma)$ is finitely generated and free the Künneth formula applies and

$$H^*(\Sigma^k) \cong H^*(\Sigma)^{\otimes k}.$$

The cohomology ring of $\mathrm{Sym}^k\Sigma$ is given by Macdonald [8] and Seroul [10]. In addition Seroul's paper [11] gives a sketch of his argument. Macdonald uses methods from algebraic geometry to give generators and relations for $H^*(\mathrm{Sym}^k\Sigma; K)$ over a field K of characteristic zero, and to show that $H^*(\mathrm{Sym}^k\Sigma)$ is torsion free. He then states incorrectly that this implies the same elements generate over the integers. Seroul confirms Macdonald's answer, using purely algebraic-topological techniques to find $H^*(\mathrm{Sym}^k\Sigma; \mathbf{Z})$ directly. In part the result may be stated as follows; we omit the statement of the relations as we will do all ring multiplication in $H^*(\mathrm{Sym}^k\Sigma)$.

Theorem 8 (Macdonald [8] and Seroul [10, 11]) *The map*

$$q_k^*\colon H^*(\mathrm{Sym}^k\Sigma; R) \to H^*(\Sigma^k; R)$$

is an isomorphism of $H^*(\mathrm{Sym}^k\Sigma; R)$ onto $H^*(\Sigma^k; R)^{S_k}$, the subring of cohomology fixed by S_k, for R a field of characteristic zero, and is injective for $R = \mathbf{Z}$. $H^*(\mathrm{Sym}^k\Sigma; \mathbf{Z})$ is generated by elements $\xi_1, \ldots, \xi_{2g}$ in degree 1 and η in degree 2 such that

$$q_k^*\xi_i = \sum_{j=1}^k \pi_j^*\alpha_i, \qquad\qquad q_k^*\eta = \sum_{j=1}^k \pi_j^*\beta,$$

where $\pi_j \colon \Sigma^k \to \Sigma$ is projection on the jth factor. A basis for $H^r(\mathrm{Sym}^k\Sigma; \mathbf{Z})$ is given by the monomials $\xi_{i_1} \cdots \xi_{i_m}\eta^n$ for which $m + 2n = r$, $i_1 < \cdots < i_m$, and $m \leq \min\{r, 2k - r\}$.

We remark that q_k is a degree $k!$ map, so q_k^* is certainly not onto $H^*(\Sigma^k)^{S_k}$ with integer co-efficients.

To avoid confusion we give different names to the generators of $H^*(\mathrm{Sym}^2\Sigma)$ and $H^*(\mathrm{Sym}^3\Sigma)$. Let

$$\zeta_i = \alpha_i \otimes 1 + 1 \otimes \alpha_i,$$
$$\theta = \beta \otimes 1 + 1 \otimes \beta,$$
$$\xi_i = \alpha_i \otimes 1 \otimes 1 + 1 \otimes \alpha_i \otimes 1 + 1 \otimes 1 \otimes \alpha_i,$$
$$\eta = \beta \otimes 1 \otimes 1 + 1 \otimes \beta \otimes 1 + 1 \otimes 1 \otimes \beta.$$

Since q_k^* is injective we shall abuse notation and not take care to distinguish between elements of $H^*(\mathrm{Sym}^k\Sigma)$ and their images in $H^*(\Sigma^k)$, and will regard the ζ_i and θ as generators of $H^*(\mathrm{Sym}^2\Sigma)$, and the ξ_i and η as generators of $H^*(\mathrm{Sym}^3\Sigma)$.

6.3 The cohomology calculation

Returning to the Mayer-Vietoris sequence, letting

$$\Phi_i \colon H^i(\mathrm{Sym}^3\Sigma) \oplus H^i(\mathrm{Sym}^2\Sigma) \to H^i(\Sigma \times \Sigma)$$

be the map $\iota^* \oplus q_2^* = (q_3 \circ \mathrm{id} \times \Delta)^* \oplus q_2^*$ we have the short exact sequence

$$0 \to \mathrm{coker}\,\Phi_{i-1} \to H^i(E_3\Sigma) \to \ker \Phi_i \to 0.$$

Since $H^i(\mathrm{Sym}^3\Sigma) \oplus H^i(\mathrm{Sym}^2\Sigma)$ is free the kernel of Φ_i is too, so the sequence splits and we get

$$H^i(E_3\Sigma) \cong \mathrm{coker}\,\Phi_{i-1} \oplus \ker \Phi_i. \qquad (6.1)$$

In what follows we calculate the kernel and cokernel of each Φ_i.

Dimension one

Both $H^1(\mathrm{Sym}^3\Sigma) \oplus H^1(\mathrm{Sym}^2\Sigma)$ and $H^1(\Sigma \times \Sigma)$ have rank $4g$, with bases $\{\xi_i\} \cup \{\zeta_i\}$ and $\{\alpha_i \otimes 1\} \cup \{1 \otimes \alpha_i\}$ respectively. For $\alpha \in H^j(\Sigma)$ we have

$$(\mathrm{id} \times \Delta)^*(1 \otimes 1 \otimes \alpha) = (\mathrm{id} \times \Delta)^* \pi_3^* \alpha$$
$$= (\pi_3 \circ \mathrm{id} \times \Delta)^* \alpha$$
$$= \pi_2^* \alpha = 1 \otimes \alpha,$$

and similarly $(\mathrm{id} \times \Delta)^*(1 \otimes \alpha \otimes 1) = 1 \otimes \alpha$, $(\mathrm{id} \times \Delta)^*(\alpha \otimes 1 \otimes 1) = \alpha \otimes 1$. Consequently

$$\Phi_1(\xi_i) = (\mathrm{id} \times \Delta)^*(\alpha_i \otimes 1 \otimes 1 + 1 \otimes \alpha_i \otimes 1 + 1 \otimes 1 \otimes \alpha_i)$$
$$= \alpha_i \otimes 1 + 2 \otimes \alpha_i.$$

Since $\Phi_1(\zeta_i) = q_2^* \zeta_i = \alpha_i \otimes 1 + 1 \otimes \alpha_i$ and $\det \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = -1$, Φ_1 maps $\mathrm{span}\{\xi_i, \zeta_i\}$ isomorphically onto $\mathrm{span}\{\alpha_i \otimes 1, 1 \otimes \alpha_i\}$. Thus

$$\ker \Phi_1 \cong \mathrm{coker}\, \Phi_1 \cong \{0\}.$$

Dimension two

$H^2(\mathrm{Sym}^3\Sigma) \oplus H^2(\mathrm{Sym}^2\Sigma)$ has basis

$$\{\xi_i \xi_j | i < j\} \cup \{\zeta_i \zeta_j | i < j\} \cup \{\eta, \theta\}$$

and rank $2\binom{2g}{2} + 2$, while $H^2(\Sigma \times \Sigma)$ has basis

$$\{\alpha_i \otimes \alpha_j\} \cup \{\beta \otimes 1, 1 \otimes \beta\}$$

and rank $4g^2 + 2$. Under Φ_2 we have

$$\xi_i \xi_j \mapsto (\alpha_i \otimes 1 + 2 \otimes \alpha_i)(\alpha_j \otimes 1 + 2 \otimes \alpha_j)$$
$$= \begin{cases} 2(\alpha_i \otimes \alpha_j - \alpha_j \otimes \alpha_i) & |i - j| \neq g, \\ 2(\alpha_i \otimes \alpha_j - \alpha_j \otimes \alpha_i) + \beta \otimes 1 + 4 \otimes \beta & j = i + g, \end{cases}$$
$$\zeta_i \zeta_j \mapsto (\alpha_i \otimes 1 + 1 \otimes \alpha_i)(\alpha_j \otimes 1 + 1 \otimes \alpha_j)$$
$$= \begin{cases} \alpha_i \otimes \alpha_j - \alpha_j \otimes \alpha_i & |i - j| \neq g, \\ \alpha_i \otimes \alpha_j - \alpha_j \otimes \alpha_i + \beta \otimes 1 + 1 \otimes \beta & j = i + g, \end{cases}$$
$$\eta \mapsto \beta \otimes 1 + 2 \otimes \beta,$$
$$\theta \mapsto \beta \otimes 1 + 1 \otimes \beta.$$

Clearly the image of Φ_2 is the span of

$$\{\beta \otimes 1, 1 \otimes \beta\} \cup \{\alpha_i \otimes \alpha_j - \alpha_j \otimes \alpha_i | i < j\}, \qquad (6.2)$$

a subspace of rank $\binom{2g}{2} + 2$. Thus the kernel of Φ_2 has rank $\binom{2g}{2}$. The set in (6.2) may be augmented to a basis for $H^2(\Sigma \times \Sigma)$, so the cokernel of Φ_2 is free of rank $4g^2 + 2 - \binom{2g}{2} - 2 = \binom{2g}{2} + 2g$. Hence

$$\ker \Phi_2 \cong \mathbf{Z}^{\binom{2g}{2}}, \qquad\qquad \operatorname{coker} \Phi_2 \cong \mathbf{Z}^{\binom{2g}{2}+2g}.$$

Dimension three

A basis for $H^3(\operatorname{Sym}^3 \Sigma) \oplus H^3(\operatorname{Sym}^2 \Sigma)$ is given by

$$\{\xi_i \xi_j \xi_k | i < j < k\} \cup \{\xi_i \eta\} \cup \{\zeta_i \theta\}.$$

If the genus of Σ is greater than one the rank is $\binom{2g}{3} + 4g$, but in genus equal to one there are only two distinct ξ_i, so the leftmost set in this union is empty and the rank of $H^3(\operatorname{Sym}^3 \Sigma) \oplus H^3(\operatorname{Sym}^2 \Sigma)$ is $4g = 4$. In either case $H^3(\Sigma \times \Sigma)$ has basis

$$\{\alpha_i \otimes \beta\} \cup \{\beta \otimes \alpha_i\}$$

and rank $4g$. We have

$$\begin{aligned}
\zeta_i \theta &\mapsto (\alpha_i \otimes 1 + 1 \otimes \alpha_i)(\beta \otimes 1 + 1 \otimes \beta) \\
&= \alpha_i \otimes \beta + \beta \otimes \alpha_i, \\
\xi_i \eta &\mapsto (\alpha_i \otimes 1 + 2 \otimes \alpha_i)(\beta \otimes 1 + 2 \otimes \beta) \\
&= 2(\alpha_i \otimes \beta + \beta \otimes \alpha_i),
\end{aligned}$$

and in genus one it follows that the kernel and cokernel of Φ_3 both have rank two. When $g \geq 2$ the triple product $\xi_i \xi_j \xi_k$ maps to 0 if i, j, k are distinct mod g, while

$$\begin{aligned}
\xi_i \xi_{i+g} \xi_j &\mapsto (2(\alpha_i \otimes \alpha_{i+g} - \alpha_{i+g} \otimes \alpha_i) + \beta \otimes 1 + 4 \otimes \beta)(\alpha_j \otimes 1 + 2 \otimes \alpha_j) \\
&= 2\beta \otimes \alpha_j + 4\alpha_j \otimes \beta
\end{aligned}$$

for $i \neq j \neq i + g$. Considering the images of $\zeta_i \theta$ and $\xi_i \xi_{i+g} \xi_j$ we see that the image of Φ_3 has rank $4g$ and that

$$\operatorname{coker} \Phi_3 \cong \frac{\operatorname{span}\{\beta \otimes \alpha_j + 2\alpha_j \otimes \beta\}}{\operatorname{span}\{2(\beta \otimes \alpha_j + 2\alpha_j \otimes \beta)\}} \cong [\mathbf{Z}/2\mathbf{Z}]^{2g},$$

so that

$$\ker \Phi_3 \cong \begin{cases} \mathbf{Z}^2 & g = 1, \\ \mathbf{Z}^{\binom{2g}{3}} & g \neq 1, \end{cases} \qquad \operatorname{coker} \Phi_3 \cong \begin{cases} \mathbf{Z}^2 & g = 1, \\ [\mathbf{Z}/2\mathbf{Z}]^{2g} & g \neq 1. \end{cases}$$

Dimension four

$H^4(\mathrm{Sym}^3\Sigma) \oplus H^4(\mathrm{Sym}^2\Sigma)$ has rank $\binom{2g}{2} + 2$ and basis
$$\{\xi_i\xi_j\eta | i < j\} \cup \{\eta^2, \theta^2\},$$
while $H^4(\Sigma \times \Sigma)$ has rank one and basis $\{\beta \otimes \beta\}$. Under Φ_4 we have
$$\begin{aligned}
\theta^2 &\mapsto (\beta \otimes 1 + 1 \otimes \beta)^2 \\
&= 2\beta \otimes \beta, \\
\eta^2 &\mapsto (\beta \otimes 1 + 2 \otimes \beta)^2 \\
&= 4\beta \otimes \beta, \\
\xi_i\xi_j\eta &\mapsto 2(\alpha_i \otimes \beta + \beta \otimes \alpha_i)(\alpha_j \otimes 1 + 2 \otimes \alpha_j) \\
&= \begin{cases} 0 & |j - i| \neq g, \\ 6\beta \otimes \beta & j = i + g. \end{cases}
\end{aligned}$$
Clearly
$$\ker \Phi_4 \cong \mathbf{Z}^{\binom{2g}{2}+1}, \qquad\qquad \mathrm{coker}\,\Phi_4 \cong \mathbf{Z}/2\mathbf{Z}.$$

Dimensions five and six

$\Sigma \times \Sigma$ and $\mathrm{Sym}^2\Sigma$ have no cohomology in dimensions five and six so the cokernel of Φ_i is trivial and the kernel is $H^i(\mathrm{Sym}^3\Sigma)$ for $i = 5, 6$. $H^5(\mathrm{Sym}^3\Sigma) = \mathrm{span}\{\xi_i\eta^2\}$ has rank $2g$ and $H^6(\mathrm{Sym}^3\Sigma) = \mathrm{span}\{\eta^3\}$ has rank one, so
$$\begin{aligned}
\ker \Phi_5 &\cong \mathbf{Z}^{2g}, & \mathrm{coker}\,\Phi_5 &\cong \{0\}, \\
\ker \Phi_6 &\cong \mathbf{Z}, & \mathrm{coker}\,\Phi_6 &\cong \{0\}.
\end{aligned}$$

Completing the proof of Theorem 5

Putting the kernels and cokernels calculated above together using equation (6.1) gives the table in Theorem 5. Taking alternating sums of Betti numbers gives
$$\begin{aligned}
\chi(\exp_3\Sigma) &= 3 - 4g + \binom{2g}{2} - \binom{2g}{3} \\
&= \frac{-4g^3 + 12g^2 - 17g + 9}{3}
\end{aligned} \tag{6.3}$$
for $g \geq 2$, and direct substitution shows it holds for $g \geq 0$ also. As a check we calculate the Euler characteristic using
$$\chi(E_3\Sigma) = \chi(\mathrm{Sym}^3\Sigma) + \chi(\mathrm{Sym}^2\Sigma) - \chi(\Sigma \times \Sigma).$$

Macdonald gives $\chi(\mathrm{Sym}^n \Sigma) = (-1)^n \binom{2g-2}{n}$, so

$$\chi(E_3 \Sigma) = -\binom{2g-2}{3} + \binom{2g-2}{2} - (2-2g)^2$$
$$= \frac{-4g^3 + 12g^2 - 17g + 9}{3},$$

in agreement with (6.3).

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